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Twodimensional Tensor Analysis Without Coördinates.

By G. Y. Rainich.

In this paper and in the papers which will follow it is intended to show how the geometry of curved space can be introduced without the use of coördinates, by a simple vector method.

When we study ordinary Euclidean space with the aid of coördinates the relation between the object of our study and the method is clear. We have a knowledge of the things which we are studying independent of the representation of them; it is therefore easy to distinguish in the representation two kinds of properties: the properties which belong to the objects of study themselves and the properties of the particular representation. We have different representations; we consider the passage from one representation to another, the transformations of coördinates; and we know that only those properties belong to the objects themselves which are not destroyed by such transformations, which are invariants.

Matters are different when we study curved space. We have no previous knowledge of the things we are studying; they are first introduced by the aid of representation. Of course we endeavour, by applying the same device of considering only properties which are not destroyed by transformations, to find out the properties of the things represented and to eliminate notions which are due only to the system of representation. But this seems to be very difficult to do when the things themselves are not familiar to us; of course, we succeed in finding out the *numbers* which have absolute significance, the invariants, but the vectors (as well as more complicated things) are not so fortunate: even now mathematicians often speak of covariant and contravariant vectors as if they were two different kinds of vectors and not only two different kinds of *representation* of vectors, as is really the case.

This shows that it would be *desirable* to treat the geometry of curved space *directly,* not using any coördinates in the development of the general theory and confining their use to the treatment of special questions. After the general theory had been established, there is no danger of losing hold on the objects of our study; it can not be denied that there are some cases in which the use of coördinates is indispensable.

I hope to show that this direct treatment of geometry is *possible* with the aid of a few simple notions of vector analysis. It must be confessed,

however, that even these very simple notions cannot be supposed to be a common property of mathematicians; and this is a great handicap for the exposition of the theory. The notion of coördinates is very deeply rooted in the modern mind, and even the notions of the theory of vectors, when they are present, are often based on the conception of coördinates. But for the understanding of the theory which I have in mind it is necessary to get rid to a certain extent of this conception. That is why it seems more appropriate to proceed gradually. In this first paper, only surfaces in ordinary space are treated, because it would make the difficulties greater if we were deprived in the beginning both of coördinates and of the help of geometrical intuition. Besides the theory of surfaces is the model on which all the higher theories are built and must be built, and it is well to master it completely before attempting generalizations.

Even here, in the theory of surfaces, it seems necessary to make a compromise with present-day custom, and introduce coördinates in order to show how it is possible to get rid of them, and even in the advanced portions of the paper to use them as an illustration and as a means of establishing the connection with the usual presentation of the theory.

The literature of both the general tensor analysis and the theory of surfaces is too large to be cited here. The development of the subject after Gauss and Riemann is largely due to Christoffel, Ricci, Levi-Civita, Weyl and Schouten. I will only name the following works which come particularly near to the treatment of the subject in this paper: C. Burali-Forti, "Fondamenti per la Geometria Differenziale su di una superficie col metodo differenziale assoluto," *Rend. Palermo,* Vol. 33 (1912_1), pp. 1-40; Edwin B. Wilson and C. L. E. Moore, "Surfaces in Hyperspace," *Proc. Amer. Acad. of Arts and Sciences* (Boston), Vol. 52, No. 6, Nov. 1916; E. H. Neville, "Multilinear Functions of Direction," Cambridge, 1921.

The contents of this paper, with the exception of the proof of the Codazzi theorem of § 10, are contained in a brief outline in my paper "Tensor Analysis Without Coördinates," *Proc. of the Nat. Acad. of Sciences,* June 1923; the main ideas were introduced in an "Essay on the Natural Geometry Of Curved Space" printed (in Russian) in the *Memoirs of the Higher Schools of Odessa,* Vol. 1, 1921.

§ 1. *The differential of a function of one variable.*

We start with the consideration of an ordinary function $\phi(\xi)$ of a real variable. The expression

1.1 $$\phi(\xi_0) + \phi'(\xi_0).(\xi - \xi_0)$$

is very important: it can be considered as an approximation to the function $\phi(\xi)$ in the vicinity of the value ξ_0, or, if the function is interpreted to represent a curve, the expression 1.1 can be considered to represent the the tangent to this curve; we shall adopt the latter point of view. The second term of our expression is nothing else, apart from the notation, than the differential of $\phi(\xi)$ for the value ξ_0 of the independent variable ξ because it is the derivative multiplied by an arbitrary quantity. We shall have to consider more general functions than $\phi(\xi)$, and together with them we shall consider certain expressions which are generalizations of $\phi'(\xi_0).(\xi - \xi_0)$; but we shall endeavour to study them from a purely geometrical point of view, without coördinates; it will be useful, therefore, to show how this point of view is applied in this simple case, which is the type of the more complicated cases. We consider the axis of abscissas and the curve represented by $\phi(\xi)$; but instead of designating the points of this axis by letters which stand for numbers giving the abscissas, we shall simply denote these points by capital Roman letters, as we do in elementary geometry. We shall write now $\phi(A)$ for the length of the perpendicular which we have to erect at the point A of the axis if we want to reach the curve. We shall, farther, use small Roman letters to designate directed segments or *vectors* of our axis; not the distances or the lengths of segments but the segments themselves. If A is the initial point of such a segment or vector h and B its end point, we shall write

$$B - A = h, \quad B = A + h; \quad \text{also } A - B = -h, \quad A = B - h;$$

by $\lambda h (\lambda > 0)$ we shall understand the vector which has the same direction (from left to right or from right to left) as h and is λ times as long. We shall use Greek letters to denote numbers or scalars. In this notation, we can write for the above representation of the tangent:

1.2 $$\phi(A) + \phi'(A).h;$$

and we may add that this is a number which gives us the length of the perpendicular to the axis which we have to erect at the point $A + h$ if we want to reach the tangent touching our curve at the point which lies over the point A of the axis. As before, the second term $\phi'(A).h$ is the differential of our function at the point A, h being the differential of the independent variable. Till now we considered the differential as the product of the derivative by the differential of the independent variable; it is important to note that the differential can be obtained directly, independently from the derivative, as the limit of the expression

1.3 $$\frac{\phi(A+\lambda h)-\phi(A)}{\lambda}$$

for λ tending toward 0. To show that the limit of 1.3 is equal to $\phi'(A).h$ we would have, interpreting h as a number, to multiply the denominator of 1.3 by h; passing to the limit we see that the derivative $\phi'(A)$ is equal to $1/h$ times the limit of 1.3 q. e. d.

§ 2. *The differential of a point function in the plane.*

We shall now apply the same notation to the case of a function $\phi(\xi,\eta)$ of two variables interpreted as representing a surface over a plane Π which shall play here the same rôle as the axis played before. We shall designate the points of this plane by capital Roman letters and we shall write as before $\phi(A)$ for the length of the perpendicular which has to be erected at the point A in order to reach the surface; we shall use small Roman letters for vectors of the plane Π, and we shall write for the vector h whose initial point is A and whose end point is B: $h=B-A$; we shall consider that a vector does not change under an arbitrary translation in the plane, so that we can bring the initial point of a vector to coincide with an arbitrary point of the plane; under $A+h$ we shall understand the point in which the end point of h falls when its initial point is brought to coincide with A; the multiplication by a number λ means the multiplication of the length of a vector without changing its direction. Vectors can be added in the usual way like forces or velocities. If axes are introduced, vectors can be *represented* by their components, but it is necessary to bear in mind that the vectors are to be conceived geometrically.

In the old notation the representation of the tangent plane is

2.1 $$\phi(\xi_0,\eta_0)+\phi_1(\xi_0,\eta_0).(\xi-\xi_0)+\phi_2(\xi_0,\eta_0).(\eta-\eta_0)$$

ϕ_1 and ϕ_2 designating the derivatives of ϕ with respect to the first and second arguments respectively. If we denote the point of the coördinates ξ_0, η_0 by A and the point of the coördinates ξ, η by B, the differences $\xi-\xi_0$, $\eta-\eta_0$ are the components of the vector $h=B-A$ which plays here the rôle of the differential of the independent variable; the sum

2.2 $$\phi_1(\xi_0,\eta_0).(\xi-\xi_0)+\phi_2(\xi_0,\eta_0).(\eta-\eta_0)$$

is considered as the differential of the function $\phi(x,y)$ in the point ξ_0,η_0. It is very important that this differential can in this case be found as the same limit as before, viz.

2.3 $$\lim_{\lambda\to 0}\frac{\phi(A+\lambda h)-\phi(A)}{\lambda}$$

This is obvious when one of the components of h is zero, i. e. either $\xi_0 = \xi$ or $\eta_0 = \eta$, and the general case can be reduced to these two.

The expression 2.3, which gives us the differential, is a function of two variables A and h, and we shall denote it with $\phi'(A, h)$. It can depend on A in any way; but it depends in a very simple way on h; in the case of one variable, it was simply the product of h into a number; it can be interpreted as a product also in our present case. If we introduce a vector g with the components $\phi_1(\xi_0, \eta_0), \phi_2(\xi_0, \eta_0)$, the expression 2.2 is the sum of the products of the corresponding components of the vectors g and h and is, therefore, equal to their (scalar) product (geometrically, the product of two vectors is the product of their lengths into the cosine of their angle). The product of two vectors h and g shall be denoted by $h.g$ or hg, so that we can write, for 2.1 in our notation

2.4 $$\phi(A) + \phi'(A, h) = \phi(A) + g.h.$$

This represents the length of the perpendicular to the plane Π at the point $A + h$ reaching the tangent plane which touches the surface in A.

§3. *Linear functions. Tensors.*

The expression 2.3 gives us the differential of a function not only in the simple cases considered above, when it can be put in the form of a product, but also in more complicated cases (e. g. in the case of a vector function) when it is not possible to present it in this form; it is therefore necessary to characterize the mode of its dependence on h in a more general way, not by its form but by its properties, and this we can do by saying that $\phi'(A, h)$ depends upon h *linearly;* we understand by a *linear dependence* or a *linear function* a function $\phi(x)$ which is characterized by the condition *

3.1 $$\phi(x + y) = \phi(x) + \phi(y).$$

For a linear function we always have

3.2 $$\phi(\lambda x) = \lambda\phi(x) \text{ and } \phi(0) = 0.$$

By the preceding we are led to the consideration of linear functions, the arguments of which are vectors. We therefore shall put here together some of their properties which we shall have to use in what follows.

1. A scalar linear function $\phi(x)$ (as the example considered above shows) is the product of its argument into a constant vector

* We suppose here that our function is continuous; if continuity is not presupposed, we must add the supplementary condition $\phi(\lambda x) = \lambda\phi(x)$.

3.4 $$\phi(x) = g.x.$$

2. Next we consider a scalar function $\phi(x, y)$ which depends linearly on each of *two* variable vectors; it can be according to 1. presented as the product of one of its arguments, say x, by a vector g, depending on the other; the dependence of g on y is linear, so that $g(y)$ is a linear vector function. We can say: *A scalar bilinear function can be presented as the product of one of its arguments by a linear vector function of the other*

3.5 $$\phi(x, y) = x.g(y)$$

3. Conversely if we have a linear vector function $g(y)$, we can multiply it by another vector x and obtain the corresponding bilinear scalar function.

4. A linear vector function $g(y)$ is called symmetrical (or self-conjugate) if the corresponding bilinear scalar function is symmetrical in the two variables, so that we have in this case $x.g(y) = y.g(x)$. The symmetrical l. v. f. has the following property: there exist two perpendicular directions and two numbers λ and μ such that, if a vector x lies in the first direction $g(x) = \lambda x$, and for a vector y of the second direction, $g(y) = \mu y$.

The symmetrical l. v. f. (or the corresponding bilinear scalar function, which is but another form of the same thing) is a very definite geometrical object; it can be visualized as a transformation which assigns a vector $g(y)$ to every given vector y (it is better to conceive in this case all the vectors as having a common initial point; then the linearity of the function finds its expression in the fact that the transformation assigns to every three vectors x, y, z, having their end points on a straight line three vectors $g(x)$, $g(y)$, $g(z)$ which also have their end points on a straight line); or as two perpendicular directions with two numbers λ and μ; or as the central conic having its axes in the directions mentioned above and the lengths of the axes being 2λ and 2μ; or, mechanically, as the stresses or strains in a thin plate. In all these cases we can, of course, introduce coördinates and *represent* things with their aid, but it is very important that the things (the l. v. f. or the bilinear scalar function) exist and can be defined and considered independently of any system of coördinates. (Compare what was said on this subject in the introduction). If we keep this in mind, there can be no harm in showing how our objects can be represented with the use of coördinates.

5. If we represent a vector x in any rectangular system of rectilinear coördinates by its components x_1, x_2 the scalar linear function will be represented by an expression

3.6 $$a_1x_1 + a_2x_2,$$

the bilinear scalar function will be represented by

3.7 $$a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2$$

and the two components of the corresponding l. v. f. will be

3.8 $$a_{11}x_1 + a_{12}x_2, \quad a_{12}x_1 + a_{22}x_2$$

6. If we have a symmetrical function then $a_{12} = a_{21}$. If, in this case, we use as axes the two perpendicular directions mentioned under 4. the expressions (3.7) and (3.8) are reduced to

3.9 $$\lambda x_1y_1 + \mu x_2y_2, \qquad \lambda x_1, \ \mu x_2$$

We will consider also multilinear functions, that is functions which depend linearly on each of several vector arguments. A scalar multilinear function will be called a *tensor,* and the number of its arguments, its *rank.* Since this notion of tensor is fundamental we state once more explicitly.

A tensor of rank r is a function whose values are numbers and whose r arguments are vectors; it is a linear function with respect to each of the arguments so that

$$\phi(x + x', y, \cdots) = \phi(x, y, \cdots) + \phi(x', y, \cdots),$$
$$\phi(x, y + y', \cdots) = \phi(x, y, \cdots) + \phi(x, y', \cdots) \text{ etc.}$$

A l. v. f. of $r - 1$ variables multiplied by an r-th variable vector gives a tensor of rank r; inversely, a tensor can be presented as the product of one of its variables by a l. v. f. of the remaining variables. A l. v. f. of $r - 1$ vector arguments is therefore but another form of a tensor of the rank r.

This definition or conception of a tensor covers essentially the usual conception. It may appear strange that we consider here a tensor as a function having numerical values; but we have to remember that, not these values but their dependence on the vector arguments, constitutes the tensor and, on the other hand, in the usual representation the components are also numbers. If we represent e. g. a tensor $\phi(x, y)$ whose expression in a particular system of coördinates is given by 3.7 by its coefficients a_{11}, a_{12}, a_{21}, a_{22}, we actually give the values of $\phi(x, y)$ for special values of the vector arguments, namely for $x = (1, 0)$, $y = (1, 0)$; $x = (1, 0)$, $y = (0, 1)$; $x = (0, 1)$, $y = (1, 0)$; $x = (0, 1)$, $y = (0, 1)$. Owing to the linearity of the function that suffices to determine it for all values of the arguments. When we have to deal with a particular problem and have our axes *actually* given, the above procedure may be useful and even necessary. But dealing with general properties, there is no advantage in giving a preference to these four sets of values of our

arguments, and we therefore consider the function $\phi(x, y)$ as known for all possible sets; in general tensor analysis, in its usual form, when we consider the components as given in any system of coördinates, we actually do the same thing, because these components are precisely different values of the function $\phi(x, y)$.

§ 4. *Differentiation of tensorfields in the plane.*

A differential $\phi'(A, h)$ of a scalar function as defined before is a linear scalar function or, as we may say now, a tensor of the first rank as far as dependence on h goes. But it depends also on A. We may say that it is a tensor for every point A; we shall call such a function a tensorfield. In general we shall call a function $\phi(A, x, y, \cdots)$ with one point argument A and several vector arguments $x, y, \cdots$ a *tensorfield*, if for every fixed value of A it is a tensor, and the number of its vector arguments we shall call its *rank*. It will be convenient for us to call a scalar function of a point of the plane such as the function $\phi(A)$ considered in § 2 *a tensorfield of the rank zero*. Adopting this terminology, we can state our former result thus: the differential of a tensorfield of the rank zero is a tensorfield of the first rank; it can be obtained with the aid of the expression *

$$4.1 \qquad \phi'(A, h) = \lim_{\lambda \to 0} \frac{\phi(B) - \phi(A)}{\lambda} \quad \text{with } B = A + \lambda h$$

Giving to the vector arguments $x, y, \cdots$ of a tensorfield $\phi(A, x, y, \cdots)$ fixed values we obtain a field of the rank zero. If we form the differential of this field according to the above definition and restore afterwards the variability to the original vector arguments we obtain a function $\phi'(A, x, y, \cdots, h)$ which has an additional vector argument, the differential of A, and since it depends on this argument also linearly we have a tensorfield of rank $r + 1$ (r being the rank of the original tensorfield). It is natural to call this ten-

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* This is not a definition of the differential: strictly speaking from the existence of the limit it does not follow that the function $\phi'(A, h)$ depends linearly on h. We could say that the differential is a linear function of h which is equal to the above limit (and if the limit is not a linear function then the differential just does not exist); or we could adopt as the definition of the differential the following expression which, if it exists, gives a linear function of h:

$$4.2 \qquad \lim_{\nu \to \infty} \frac{\phi(B\nu) - \phi(C\nu)}{\lambda_\nu} \quad \text{with } B\nu \to A,\ C\nu \to A,\ \lambda_\nu \to 0 \text{ and } \frac{B\nu - C\nu}{\lambda_\nu} \to h.$$

But we will not dwell here any longer on these subtleties because they would mar our principal point at issue.

sorfield the differential of the tensorfield $\phi(A, x, y \cdots)$. Differentiation thus raises the rank of a field by unity.

Let us start now with a field of rank zero $\phi(A)$; we form its differential $\phi'(A, h)$; this is a field of rank one; we form its differential, which will be a field of rank two $\phi''(A, h, k)$. It is to be noticed that although both vectors h and k are differentials of the independent variable A we assume them as *different* and independent, one from the other; in the usual notation, these different differentials of the same independent variable are sometimes designated as dx and δx.

In what follows we will have to use the fact that a second differential is a symmetrical function of its two vector arguments, which are the differentials of the independent variable:

4.3 $$\phi''(A, x, y, \cdots, h, k) = \phi''(A, x, y, \cdots, k, h)$$

This follows from the fact that both are the limits of the expression:

$$\frac{\phi(A + \lambda h + \mu k) - \phi(A + \lambda h) - \phi(A + \mu k) + \phi(A)}{\lambda\mu}$$

for $\lambda \to 0$, $\mu \to 0$, if we suppose that it is permitted to change the order of passing to the limit. Expressed in coördinates, this property is equivalent to the changing of the order of differentiation.

§ 5. *Tensors belonging to a surface.*

We pass now at last to the consideration of a surface. We suppose that it has a definite tangent plane at every point. We will consider the vectors of a tangent plane to the surface as vectors "belonging" to the surface. They play, in what follows, the rôle of what are sometimes called infinitesimal vectors of the surface (or differentials) but we prefer to deal with finite quantities and consider limits instead of infinitesimals. It will be convenient for us in most cases to visualize the vectors of a definite tangent plane as having their initial points in the point of contact A of this plane with the surface, and we will call the totality of these vectors *the bundle A;* but we have the right, when necessary, to move them freely in the plane (without turning them). The vectors of the bundle A will be denoted by x_A, y_A, $\cdots$. A multilinear function, $\phi(x_A, y_A, \cdots)$ whose arguments are vectors of the bundle A, will be called a tensor of the bundle A, and we will say that such a tensor belongs to the surface. All bundles in themselves are alike, the only difference being in their location in space; therefore all that was said about individual tensors (not tensorfields) applies immediately to tensors of every bundle—the

Algebra of tensors belonging to a surface is the same as the Algebra of tensors in the plane. But things change when we come to the Analysis of tensors, to the tensorfields. The definition is practically the same: we shall call a tensorfield a function $\phi(A, x_A, y_A, \cdots)$ with one point argument A, which is a point of the surface, and several vector arguments, which are vectors of the bundle A corresponding to the point argument; the number of the vector arguments we, of course, shall call the rank of the field. But we have to make an effort of imagination to grasp this peculiar kind of function which is a tensorfield. Take a tensor of the second rank $\phi(A, x_A, y_A)$ for instance; x_A can be every vector of the surface and the same is true of y_A; but as soon as we give a definite value to the point argument A, we have restricted our choice of values which can be attributed to the vector arguments; i. e. they must belong to the bundle A; in particular (and we shall very soon see the importance of this remark) we cannot give to the vector arguments fixed values and in this way reduce the rank of a field, because there are no equal vectors in different bundles, in general, since vectors of different bundles lie in different planes which are, in general, not parallel.

As a first example of a tensorfield let us consider the following:

5.1 $$\epsilon(A, x_A, y_A) = x_A \cdot y_A.$$

That means that in every bundle of the surface we consider the product of two vectors. In a sense this tensor is the same in all bundles, but not in the sense that it has the same values for the same arguments because there is no such thing as the same arguments in different bundles.

§ 6. *The indicator.*

We proceed to show another example of a tensorfield and at the same time to introduce a most important tensor.

We fix for the time being the point A of the surface; we agree to denote by B_A the (orthogonal) projection of the point B on the tangent plane which touches our surface at the point A; we choose a direction on the normal at A as the positive direction. Let $\delta(B_A)$ denote the distance from the point B_A to the point B (affecting it with the sign — if the direction from B_A to B corresponds to the negative direction on the normal). $\delta(B_A)$ is a tensorfield of rank zero which is defined * in the tangent plane at A (it is not yet a tensorfield which belongs to the surface). We know from § 2 (our δ

* We consider a small vicinity of the point A on the surface such that no two points of this vicinity have the same projections on the tangent plane. δ is of course only defined in a vicinity of the point A on the plane.

is the ϕ of § 2) that the perpendicular distance at the point $B_A + h$ to the tangent plane of the point B is given by

6.1 $$\delta(B_A) + \delta'(B_A, h).$$

We apply this formula for the point $B = A$; since the two planes coincide this distance must be zero for every h, i. e.

6.2 $$\delta(A) + \delta'(A, h) = 0;$$

whence we conclude that at the point A (which is the same as A_A) both the tensor δ and its first differential vanish. We consider the second differential of δ at the point A: $\delta''(A, h, k)$. The vectors h and k are vectors of the tangent plane in A or vectors of the bundle A; we can write them h_A, k_A; and the tensor $\delta''(A, h_A, k_A)$ is a tensor of the bundle A, a tensor belonging to the surface. What we did for the point A of the surface we can repeat for every other point (determining the positive direction on the normal by continuity), and we shall have then a definite tensor in every bundle; their totality constitutes a tensorfield which we can designate by $\sigma(A, h_A, k_A)$. We shall call this tensor the *indicator,* because it is equivalent to Dupin's indicatrix; we may remark also that Gauss's second differential form is the representation of this tensor, as will be seen a little later. Since at every point the indicator is introduced as the second differential of a *plane* tensorfield, it is symmetrical in its vector arguments (comp. the end of § 4). According to property 2. § 3, we can present σ as the product of one of its arguments by a linear vector function of the other. We call this l. v. f. s_A and have

6.3 $$\sigma(h_A, k_A) = h_A . s_A(k_A) = s_A(h_A) . k_A.$$

Since σ is symmetrical, s is also, according to our definition in 4. § 3, and has the two perpendicular directions and the two numbers λ and μ mentioned there.

To show the connection with the usual exposition of the theory of surfaces, let us introduce for a moment coördinate axes. We naturally choose the two perpendicular directions of the l. v. f. s_A as the ξ and η axes, and for the ζ axis we take the normal to the surface at the point A. If we designate the coördinates of the point B_A by ξ and η, and agree to write $\delta(\xi, \eta)$ for $\delta(B_A)$, the equation of our surface in the neighborhood of the point A will be

$$\zeta = \delta(\xi, \eta).$$

The value of the function δ and its first derivatives in the point A, whose coördinates are 0, 0, vanish and the second derivative $\frac{\partial^2 \delta}{\delta\xi\partial\eta}$ is zero. Gauss's

first differential form is simply $d\xi^2 + d\eta^2$ and it is easy to see that in this case the second form is $\lambda d\xi^2 + \mu d\eta^2$. Since this coincides with the coördinate expression for $\sigma(h, h)$, we can say that for every point coördinates can be chosen in such a way that σ is represented by the second differential form, which justifies our remark that the second form is a representation of the tensor σ. The connection with Dupin indicatrix is also obvious in this system of coördinates.

§ 7. *Another way of introducing the indicator.*

In this section and in the following we shall introduce vectors which do not belong to our surface, and shall consider their sums and products. Since, however, the initial points of two vectors can always be brought together, this does not require any new definitions. Small Roman letters a, b, c, will designate unit normal vectors to the surface at the points which are designated by the corresponding capital letters A, B, C. Since the product of a vector by the same vector is equal to the square of its length, we shall have for a unit vector $a^2 = a.a = 1$. On the other hand, a is perpendicular to the vectors of the bundle A, so that $a.x_A = 0$ (the product of two vectors is zero if they are perpendicular, and only then). Now we observe that for a point X outside the plane tangent at A and at a perpendicular distance ξ from this plane we can write

$$X = X_A + \xi.a.$$

In particular, for the point B and for the point of the tangent plane in B whose projection is $B_A + h_A$ we can write respectively

$$B_A + a.\delta(B_A) \qquad \text{and } B_A + h_A + a.\{\delta(B_A) + \delta'(B_A, h_A)\}.$$

The vector which joins these two points is a vector of the tangent plane in B and consequently perpendicular to b; in our notation, we can express it by writing that the difference of the two expressions multiplied by b equals zero:

$$7.1 \qquad h_A.b + (ab).\delta'(B_A, h_A) = 0;$$

we can write also

$$7.2 \qquad h_A.a + (ab).\delta'(A, h_A) = 0$$

since both terms here are zero.

Subtracting (7.2) from (7.1) and dividing by λ, we have

$$h_A.\frac{b-a}{\lambda} + (ab).\frac{\delta'(B_A, h_A) - \delta'(A, h_A)}{\lambda} = 0.$$

We put now

7.3 $$B_A = A + \lambda k_A$$

and let λ tend toward zero; b tends toward a, and we have

7.31 $$h_A . \lim_{\lambda \to 0} \frac{b - a}{\lambda} + \delta''(A, h_A, k_A) = 0.$$

On the other hand we can show that the $\lim \frac{b-a}{\lambda}$ is a vector of the plane tangent in A; indeed, since a and b are both unit vectors we have why?

7.4 $$b . \frac{b-a}{\lambda} + \frac{b-a}{\lambda} . a = \frac{b^2 - a^2}{\lambda} = 0;$$

making λ tend toward zero, we see that $a . \lim \frac{b-a}{\lambda} = 0$ which establishes the point; this permits us, using 6.3, to write 7.31 in the form

7.5 $$\dot{a} = \lim_{\lambda \to 0} \frac{b-a}{\lambda} = - s_A(k_A).$$

We could have introduced the indicator by this relation but then we would have to prove that it is a symmetrical l. v. f.

§ 8. *The curvature of a curve on the surface.*

We consider a curve as a function $C(\tau)$, the argument τ of which is a scalar—the length of the arc from a fixed point—and the values of which are points of the space, or in particular of our surface, if we have a curve drawn on it. The limit

$$\lim_{\tau_1 \to \tau} \frac{C(\tau_1) - C(\tau)}{\tau_1 - \tau}$$

gives us the unit tangent vector which we designate by t. We denote by n and l the unit vectors of the principal normal and the binormal, and with a dot over a letter we denote differentiation with respect to the parameter τ; with this notation the formulas of Frenet can be written in the form

8.1 $$\dot{t} = \frac{n}{\rho}, \qquad \dot{n} = -\frac{t}{\rho} - \frac{b}{\rho_1}, \qquad l = \frac{n}{\rho_1}$$? $\dot{l} = \frac{n}{\rho_1}$

where ρ and ρ_1 are the radii of the first and second curvatures of the curve. The unit normal vector a of the surface must lie in the plane of n and l,

so that $a = \alpha n + \beta l$, and we find, on multiplying this by n, that $\alpha = an$, since $n^2 = 1$ and $nl = 0$. We have now

$$8.2 \qquad \dot{a} = \lim \frac{b - a}{\tau_1 - \tau} = -s(t) \quad \text{and} \quad \dot{a}.t = -s(t).t = -\sigma(t, t).$$

On the other hand

$$\dot{a} = \frac{\partial}{\partial \tau}(\alpha n + \beta l) = \dot{\alpha}.n + \dot{\beta} l + \alpha.\dot{n} + \beta.\dot{l}$$

and, using Frenet's formulas,

$$\dot{a} = \dot{\alpha}.n + \dot{\beta}.l - \alpha.\frac{t}{\rho} - \alpha.\frac{l}{\rho_1} + \beta.\frac{n}{\rho_1}$$

Multiplying this by t we get, since $tl = tn = 0$,

$$8.3 \qquad \dot{a}.t = -\frac{\alpha}{\rho} = -\frac{an}{\rho}.$$

Thus we have from 8.2 and 8.3

$$8.4 \qquad \frac{an}{\rho} = \sigma(t, t) \quad \text{or} \quad \frac{1}{\rho} = \frac{\sigma(t, t)}{an}.$$

This shows that, if we know the indicator, we can find the curvature of a curve drawn on the surface with the aid of its unit tangent vector and the unit vector of the principal normal. Of course, the Euler theorem and the Meusnier theorem are included in this formula. To obtain the former we notice that for a normal section, n coincides with a so that the denominator is equal to unity and we have $1/\rho = \sigma(t, t)$ or

$$1/\rho = t_1^2.\lambda + t_2^2.\mu;$$

we recognize λ and μ as the two principal curvatures of the surface and the two corresponding directions are, of course, the two principal directions. To prove the theorem of Meusnier we have to fix t and to vary n; since the lengths of both a and n are unity, their product is simply equal to the cosine of their angle which gives us the theorem of Meusnier.

§ 9. *Differentiation on the surface.*

We will try now to apply the notion of differentiation to tensorfields belonging to the surface. Our previous definitions can not be applied

directly, they require modification in two points. The necessity of the first modification can be brought out even in the simplest case of a tensorfield of the rank zero $\phi(A)$. Of course we can form the expression (4.1)

$$\frac{\phi(B)-\phi(A)}{\lambda}$$

but the connection between B and λ which was established previously with the aid of the formula $B=A+\lambda h$ has to be established in the case of tensorfields belonging to a surface in some other way, because B must be a point of the surface and this can not be if it is given by the above formula; but we shall find a way out of the difficulty by requiring that the *projection* of B on the tangent plane at A should satisfy this formula, so that the supplementary condition in our case will be * (comp 7.3)

9.1 $$B_A=A+\lambda h_A.$$

The formula 9.1 establishes a relation between the vector λh_A of the bundle A and the point B of the surface in the vicinity of this bundle A, such that to every vector of the bundle A (of sufficiently small length) there corresponds a point of the vicinity of A and vice versa. And that suffices for the definition of the differential of a tensorfield of the rank zero. But if we want to differentiate tensorfields of higher rank also, we will have to establish a relation between the vectors of a bundle A and the *vectors* of neighboring bundles. To show this let us recall how we defined the differential of a tensor $\phi(A, x_A, y_A \cdots)$ of a rank higher than zero; we obtained from ϕ a tensor of the rank zero by giving to the vector arguments fixed values and after the differentiation we restored to them their variability. But as we noticed before (comp. § 5), we cannot give to the vector arguments of a tensor belonging to the surface fixed values, that is the same value in different points, because in each point the vectors have to belong to the corresponding bundle. Let us take for instance a tensor of the first rank $\phi(A, x_A)$ and consider the ratio

$$\frac{\phi(B, x_B)-\phi(A, x_A)}{\lambda}.$$

B is a variable point, depending on λ. If we want to pass to the limit the vector x_B has also to have a definite value for every value of λ; in the previous case this was simply attained by putting x_B equal to x_A, and this was possible because both these vectors belonged to the same plane. But it is not possible

* The general definition of the differential given in the footnote to § 4 is applicable to the present case without modification.

here because the bundle B does not contain, in general, vectors which are equal to vectors of the bundle A, since these are vectors of different planes. We see that we must establish a relation between vectors of different bundles so that given a vector of the bundle A we may have fixed the corresponding vectors in each of the neighboring bundles. The most natural way to establish this correspondence seems to be to choose as the vector of the bundle B which corresponds to the vector x_A *the vector* x_B *whose projection on the plane tangent in* A *is equal to* x_A. We shall denote this vector by $x_{A \cdot B}$. This $x_{A \cdot B}$ can be found when x_A is given with the aid of the formula

$$x_{A\cdot B} = x_A - a\,\frac{x_A . b}{a . b}\;; \tag{9.2}$$

to prove it we first multiply both terms by b; we get

$$b . x_{A\cdot B} = x_A . b - (a . b)\,\frac{x_A . b}{a . b} = x_A . b - x_A . b = 0$$

and this shows that the first member of 9.2 is a vector perpendicular to b, i. e. belonging to the bundle B; on the other hand, since the difference between the vectors $x_{A\cdot B}$ and x_A, i. e. the vector which joins their end points when their initial points are brought together, is $a\dfrac{x_A \cdot b}{a . b}$, that is perpendicular to the plane tangent to the surface in A, their projections on this plane must be the same and, since x_A belongs to this plane, it must be the projection of the second.

Now we can write down our definition of the differential in the case of a tensorfield of any rank belonging to a surface; it is

$$\phi'(A, x_A, y_A, \cdots, h_A) = \lim \frac{\phi(B, x_{A\cdot B}, y_{A\cdot B}, \cdots) - \phi(A, x_A, y_A, \cdots)}{\lambda} \tag{9.3}$$

$$\text{with } B_A = A + \lambda h_A.$$

We will discuss *representations* of tensorfields, etc. in another paper; here we will state only that the differentiation just defined is the thing of which the contravariant differentiation and the covariant differentiation are but representations.

We must say that we could give to the vectors x_B in the formula 9.3 other values than those defined by formula 9.2 without changing the definition of the differential; it is only necessary that these values satisfy the condition,

$$\lim \frac{x_B - x_{B\cdot A}}{\lambda} = 0 \tag{9.4}$$

Indeed, if this condition is satisfied, the difference between the two expressions for the differential

$$\lim \frac{\phi(B, x_B) - \phi(A, x_A)}{\lambda} - \lim \frac{\phi(B, x_{B\cdot A}) - \phi(A, x_A)}{\lambda}$$

$$= \lim \frac{\phi(B, x_B - x_{B\cdot A})}{\lambda}$$

would be zero, as we can see with the aid of the relations 3.2 and 9.4.

Among the many possible definitions of x_B which satisfy the condition 9.4 we shall consider only one which we shall write $x_{A.B}$ (with the dot below)

9.5 $$x_{A.B} = x_A - b(x_A \cdot b)$$

The geometrical meaning of this vector $x_{A.B}$ is: a vector of the bundle B which is equal to the projection of x_A on the tangent plane at B; the proof of this does not differ from the proof which follows the formula 9.2.

The two vectors $x_{A\cdot B}$ and $x_{A.B}$ satisfy the remarkable relation

9.6 $$x_{A\cdot B} \cdot y_{A.B} = x_A \cdot y_A$$

which is the real source of the connection between the covariant and contravariant representations and which can be proved by the substitution of the values 9.2 and 9.5 in the left-hand member; we get, indeed,

$$\left[x_A - a \frac{x_A . b}{ab} \right] \cdot \left[y_A - b(y_A . b) \right] =$$

$$x_A . y_A - (x_A . b)(y_A . b) + \frac{ab}{ab}(x_A . b)(y_A . b) = x_A . y_A.$$

We may remark also that the two operations: the one leading from x_A to $x_{A\cdot B}$ and the other leading from x_A to $x_{A.B}$ are inverse so that

9.7 $$x_{A\cdot B.A} = x_A \quad \text{and} \quad x_{A.B\cdot A} = x_A.$$

This can be proved again by direct substitution, but it is obvious from the geometrical definitions of both operations.*

* The connection of these operations with the parallel displacement introduced by Levi-Civita is obvious. I find a very clear statement of the point of view on parallel displacement as establishing a connection between vectors of what we call different bundles in a recent article by *P. Dienes* in the 47th volume of the *Rendiconti Palermo*, p. 144.

§ 10. *Examples of differentials.*

As examples we will consider the differentials of our two fundamental tensors ϵ and σ.

According to the definitions of the tensor ϵ (5.1) and the differential (9.3), the differential of ϵ will be

$$\lim \frac{x_{A^{\cdot}B} \cdot y_{A^{\cdot}B} - x_A \cdot y_A}{\lambda}$$

But we have according to 9.2

$$x_{A^{\cdot}B} \cdot y_{A^{\cdot}B} = \left(x_A - a.\frac{x_A . b}{ab}\right)\left(y_A - a.\frac{y_A . b}{a.b}\right) = x_A . y_A - \frac{x_A . b}{ab} \cdot \frac{y_A . b}{ab}$$

so that we have to consider the limit of

$$-\frac{1}{\lambda} \cdot \frac{x_A . b}{ab} \cdot \frac{y_A . b}{ab} = -\frac{\lambda}{(ab)^2} \cdot \frac{x_A . b}{\lambda} \cdot \frac{y_A . b}{\lambda} \; ;$$

since

10.1 $$\lim \frac{x_A . b}{\lambda} = \lim \left(x_A . \frac{b-a}{\lambda}\right) = x_A \lim \frac{b-a}{\lambda} = - x_A . s_A(h_A)$$

is finite, we find that our differential is *zero;* we ought not to be astonished at that result since we have noticed already that ϵ is the same tensor in every point.

This fact that the differential of the tensorfield ϵ vanishes in every point, is of fundamental importance for the usual representation of the theory of surfaces and the theory of curved space in general.

We consider now the differential of the indicator $\sigma(A, h_A, k_A)$; we return to the equation 7.1

10.2 $$b . h_A + (ab)\delta'(B_A, h_A) = 0$$

and we write the same equation for another point C with $C_A = B_A + \lambda k_A$

10.3 $$c . h_A + (ac) . \delta'(C_A, h_A) = 0;$$

subtracting 10.3 from 10.2 and dividing by λ we obtain

10.4 $$\frac{c-b}{\lambda} . h_A + \left(a . \frac{c-b}{\lambda}\right) . \delta'(C_A, h_A) + (ab)\, \frac{\delta'(C_A, h_A) - \delta'(B_A, h_A)}{\lambda} = 0.$$

It is easy to see that when λ tends toward 0, $\dfrac{C_B - B}{\lambda}$ tends toward a vector

whose projection on the plane tangent at A is $\dfrac{C_A - B_A}{\lambda} = k_A$, that is toward $k_{A^\cdot B}$; therefore, applying 7.5 and substituting c for b and b for a we obtain

$$\lim_{\lambda \to 0} \frac{c - b}{\lambda} = - s_B(k_{A^\cdot B}).$$

The 10.4 gives now for $\lambda \to 0$

$$- s_B(k_{A^\cdot B}) . h_A + [a . s_B(k_{A^\cdot B})] . \delta'(B_A, h_A) + (ab) . \delta''(B_A, h_A, k_A) = 0;$$

we can substitute here $h_{A.B}$ for h_A in the first term because that is equivalent to subtracting $b(h_A . b)$ from the second factor and that does not change the product, b being perpendicular to the other factor. Introducing now σ instead of s (with the aid of the relation 6.3) in the first term we have

$$\text{10.5} \qquad - \sigma(B, h_{A.B}, k_{A^\cdot B}) + [a . s_B(k_{A^\cdot B})] . \delta'(B_A, h_A) \\ + (ab) . \delta''(B_A, h_A, k_A) = 0.$$

This holds for any point B; we will write this relation for $B = A$ but noticing that in this case the second factor of the second term is zero (comp. 6.2) we can substitute anything for the first factor and we shall choose the first factor the same as in 10.5

$$- \sigma(A, h_A, k_A) + [a . s_B(k_{A^\cdot B})] . \delta'(A, h_A) + (aa) . \delta''\ (A, h_A, k_A) = 0.$$

We subtract this from 10.5 and divide by λ

$$- \frac{\sigma(B, h_{A.B}, k_{A^\cdot B}) - \sigma(A, h_A, k_A)}{\lambda} + [a . s_B(k_{A^\cdot B})] \frac{\delta'(B_A, h_A) - \delta'(A, h_A)}{\lambda} \\ + \frac{ab - aa}{\lambda} \delta''(B_A, h_A, k_A) + \frac{\delta''(B_A, h_A, k_A) - \delta(A, h_A, k_A)}{\lambda} = 0.$$

We put now $B_A = A + \lambda l_A$ and let λ tend toward zero. The limit of the first member is, according to our definition, the differential of σ; the first term of the second factor tends toward 0 because $s_B(k_{A^\cdot B})$ tends toward $s_A(k_A)$ which is perpendicular to a, and the second factor is finite; in the third term the first factor tends also toward 0 (comp. 7.4); the last term tends toward the differential of δ''; we have finally

$$\text{10.6} \qquad \sigma'(A, h_A, k_A, l_A) = \delta'''(A, h_A, k_A, l_A)$$

the differential of σ is the third differential of δ in the point A; from this we conclude: *the differential of the indicator is symmetrical in all its three*

arguments, since this is true of $\delta'''(A, h_A, k_A, l_A)$ which is a differential of a *plane* tensorfield (comp. the end of § 4),

s(h_A)

$$\sigma'(A, h_A, k_A, l_A) = \sigma'(A, h_A, l_A, k_A) \quad \text{or} \quad s'(k, l) = s'(l, k). \tag{10.7}$$

In order to see what corresponds to this theorem in the usual theory of surfaces we recall (property 6, § 3) that the symmetricity of a tensor corresponds to the symmetricity of the indices in the coefficients which represent it in a system of coördinates and we conclude that the representation of the differential of the second form of Gauss (and that is its absolute differential) must be symmetrical in its indices; this is nothing else than the Codazzi theorem (comp. e. g. Bianchi, *Vorlesungen ueber Differentialgeometrie,* Vol. 1, Lpzg., 1899, p. 91). We can say thus that we have proved the Codazzi theorem in its intrinsic form.

§ 11. *The differentiation of curves on the surface.*

We introduced in § 8 the unit tangent vector to a curve practically as a derivative. Now this derivative, the vector t, is itself a function of the parameter τ and we may differentiate again. We consider the differential as the limit

$$\dot{t} = \lim_{\tau_1 \to \tau} \frac{t_B - t_A}{\tau_1 - \tau} \quad \text{with } B = C(\tau_1),\ A = C(\tau) \cdot \tag{11.1}$$

We have already used this vector $\dot{t}$ in § 8. But the consideration of the above limit implies the operation of subtraction applied to two vectors belonging to two *different* bundles in which case the operation cannot be said to be fully legitimate from the point of view of the surface (think of the inhabitants of the surface for whom there is no such thing as the threedimensional space in which the surface is imbedded; the importance of this point of view will be perhaps clearer if we consider that our threedimensional space may be curved and that we are in precisely the same position if we do not want to recur to higher spaces); the considerations of § 5 suggest that we form instead of 11.1 the limit

$$\lim_{\tau_1 \to \tau} \frac{t_{B.A} - t_A}{\tau_1 - \tau} \tag{11.2}$$

where the minuend and subtrahend belong both to the same bundle. We may refer to the operation defined by 11.1 as space differentiation and to the process defined by 11.2 as surface differentiation, and to their results as the space and the surface derivatives; the latter we will denote with t'. Substi-

tuting in 11.2 its expression for $t_{B.A}$ (comp. 9.5) we find

$$11.3 \qquad t' = \lim \frac{t_{B.A} - t_A}{\tau_1 - \tau} = \lim \frac{t_B - a(t_B.a) - t_A}{\tau_1 - \tau}$$
$$= \lim \frac{t_B - t_A}{\tau_1 - \tau} - a(\lim \frac{t_B - t_A}{\tau_1 - \tau} .a) = \dot{t} - a(\dot{t}.a).$$

The vector t' plays the same rôle on the surface as $\dot{t}$ plays in the space or on the plane; we see, first, that it belongs to the surface because it is the limit of the difference of two vectors which belong to the bundle A; it is normal to the curve because both $\dot{t}$ and a are; now, the length of $\dot{t}$ represents the curvature of the curve; it is natural to consider the length of t' as its *surface* curvature.

Let us first consider the case, which corresponds to the straight line in the plane, when this curvature, and consequently the vector t', is zero. We have $\dot{t} = a(a.\dot{t})$ and since $\dot{t}$ has the direction of the principal normal to the curve and a lies in the normal to the surface, we see that, in the case considered, the normals to the surface along the curve coincide with its principal normals; i. e. we have the geodesics.

We shall compute now the length of the vector t'. Squaring the relation 11.3 and taking into account that the length of a is unity by definition, we find

$$t'^2 = \dot{t}^2 - 2(\dot{t}a).(\dot{t}a) + (\dot{t}a)^2a^2 = \dot{t}^2 - (\dot{t}a)^2;$$

remembering that the product of two vectors is equal to the product of their lengths into the cosine of their angle, we finally obtain for our curvature

$$|\, t' \,| = |\, \dot{t} \,| \sin \psi,$$

that is: the surface curvature is the space curvature times the sine of the angle between the principal normal of the curve and the normal to the surface; but that is the property of the geodesic curvature. Our surface curvature is, therefore, the geodesic curvature of the curve.

§ 12. *The curvature and the Riemann tensor.*

The curvatures of the curves drawn on the surface are determined, according to 8.4, by the tensor σ or the corresponding linear vector function $s(h)$, or by its two characteristic numbers λ and μ. On the other hand, the curva-

ture properties of the surface are dependent upon the connection between different bundles introduced in § 9, and we shall in this last section show the relation between these two points of view. The full discussion of the question is, however, more appropriate when spaces of higher dimensionality are considered, because some relations degenerate in two dimensions and their full significance cannot be realized in this case; that is why we shall restrict ourselves here to the demonstration of the dependence mentioned above.

We consider the expression

12.1 $$R = \lim_{\lambda\to 0} \frac{x_{A.B.C.A} - x_{A.B.C.A}}{\lambda^2} \text{ with } B_A = A + \lambda h_A,\ C_A = A + \lambda k_A.$$

We have

$$x_{A.B} = x_A - b(x_A.b)$$
$$x_{A.B.C} = x_A - b(x_A.b) - c(x_A.c) + c(cb)(x_A.b).$$
$$x_{A.B.C.A} = x_A - b(x_A.b) - c(x_A.c) + c(cb)(x_A.b) + a(ab)(x_A.b) + a(ac)(x_A.c) - a(ac)(cb)(x_A.b)$$

and similarly

$$x_{A.C.B.A} = x_A - c(x_A.c) - b(x_A.b) + b(bc)(x_A.c) + a(ac)(x_A.c) + a(ab)(x_A.b) - a(ab)(bc)(x_A.c)$$

so that

$$x_{A.B.C.A} - x_{A.C.B.A} = \{(x_A.c)[b - a(ab)] - (x_A.b)[c - a(ac)]\}(bc);$$

we had before (10.1)

$$\lim \frac{x_A.b}{\lambda} = - x_A.s(h_A);$$

similarly

$$\lim \frac{x_A.c}{\lambda} = - x_A.s(k_A).$$

Then

$$\lim \frac{b - a(ab)}{\lambda} = \lim \frac{b - a + a(aa) - a(ab)}{\lambda}$$
$$= \lim \frac{b-a}{\lambda} - a(a.\lim \frac{b-a}{\lambda})$$
$$= - s_A(h_A) + a(a.s_A(h_A)) = - s_A(h_A).$$

Since both b and c tend toward a, their product tends toward unity and we obtain finally

$$R = [x.s(k)]s(h) - [x.s(h)].s(k)$$

12.2
$$= \begin{vmatrix} s(h) & s(k) \\ x.s(h) & x.s(k) \end{vmatrix}$$

where the subscript A, which must affect every letter, is dropped for the sake of brevity. This is a trilinear vector function; according to § 3 we can write it as a tensor of the fourth rank, multiplying it by an arbitrary vector y. This tensor

12.3
$$\rho(x, y, h, k) = \begin{vmatrix} s(h).y & s(k).y \\ s(h).x & s(k).x \end{vmatrix} = \begin{vmatrix} \sigma(h, y) & \sigma(k, y) \\ \sigma(h, x) & \sigma(k, x) \end{vmatrix}$$

is the Riemann tensor of the surface. Using the expressions 3.7 for a bilinear function this reduces to

12.4
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} y_1 & y_2 \\ x_1 & x_2 \end{vmatrix} \cdot \begin{vmatrix} h_1 & h_2 \\ k_1 & k_2 \end{vmatrix} = \lambda\mu \begin{vmatrix} y_1 & y_2 \\ x_1 & x_2 \end{vmatrix} \cdot \begin{vmatrix} h_1 & h_2 \\ k_1 & k_2 \end{vmatrix}$$

which proves the identity of the expression 12.1 with what is usually called the Riemann tensor.

Differentiating the expression 12.3 we easily prove, using the Codazzi theorem 10.7

12.5
$$\rho'(x, y, h, k, l) + \rho'(x, y, k, l, h) + \rho'(x, y, l, h, k) = 0$$

these are the so-called Bianchi relations, which in coördinate form are written

$$R_{ij,klm} + R_{ij,lmk} + R_{ij,mkl} = 0.$$

§ 13. *Conclusion.*

In what precedes, we often made use of points and vectors which lie outside the surface we were studying. It is important, however, to state that the fundamental objects, relations, and operations, though they were introduced with the aid of these extraneous things, exist independently of them. We can consider them after having forgotten everything that lies outside the surface. These fundamental things are: vectors and tensors belonging to the surface, the connection between the vectors of a bundle and the neighboring bundles, the connection between vectors of different bundles, differentiation, which is based upon these connections, and the Riemann tensor introduced by the formula 12.1, also with the aid of these connections. They constitute the real theory of surfaces and they should serve as the starting point for generalizations, that is for the theory of curved space. We will treat some

questions connected with this point of view in other papers. It is necessary, however, to say here a few words about a peculiarity of the two-dimensional case, which played an important part in the development of these theories.

The equation 12.3 shows that the Riemann tensor is determined when we know the indicator; but if the Riemann tensor is given, the indicator remains partly undetermined; only the product of the numbers λ and μ is known, their sum and the principal directions of s remain arbitrary. This is, as stated above, a peculiarity of the two-dimensional case: in any other case a tensor $\rho(x, y, h, k)$ connected with a tensor of the second rank $\sigma(x, y)$ by an equation of the form 12.3 would not only be determined by it but would also determine the latter completely. The fact that it is not so in the theory of surfaces leads to a natural division of the properties of surfaces in two classes: those which depend only on ρ and those which depend on the indicator. Only the first part of the theory was generalized by Riemann and was known as the theory of curved space for a long time.

Fractional Operations as Applied to a Class of Volterra Integral Equations.*

By H. T. Davis.

Introduction. The method in general use for the discussion of solutions of Volterra integral equations of the type

$$u(x) = \phi(x) + \lambda \int_c^x \frac{G(x,s)}{(x-s)^a} u(s)\,ds, \quad a < 1, \tag{1}$$

where $G(x,s)$ is a continuous function of both variables, may be briefly sketched as follows.†

By successive iteration of (1) a new equation is obtained

$$\begin{aligned} u(x) = [\phi(x) + \lambda \int_c^x K(x,s)\,\phi(s)\,ds + \cdots \\ + \lambda^{n-1} \int_c^x K^{(n-1)}(x,s)\phi(s)\,ds] + \lambda^n \int_c^x K^{(n)}(x,s)u(s)\,ds \end{aligned} \tag{2}$$

where $K^{(i+1)}(x,s)$ is the ith iterated kernel of $K(x,s) = \dfrac{G(x,s)}{(x-s)^a}$. It can be easily proved that in equation (2), if n is chosen sufficiently large, the kernel $K^{(n)}(x,s)$ will be a continuous function of both variables and, therefore, the solution of (2) can be discussed from the ordinary theory of Volterra equations. Consequently, since it may be shown that any solution of (2) is also a solution of (1), and conversely, we know that equation (1) has one and only one solution and the homogeneous equation, obtained by assuming $\phi(x) \equiv 0$, has no solution except the trivial one.

The more general case where $G(x,s)$ may have infinite discontinuities in x alone presents a problem of some difficulty and the general existence theorem does not appear to have been given.‡

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* Presented to the Chicago Section of the American Mathematical Society, April 13, 1923.

Since the present article was written there has appeared a paper by Paul Levy in the *Bulletin des Sciences Mathématiques*, Vol. 47 (Sept. and Oct., 1923), pp. 307 and 343 on the same subject. While the discussion of fractional operations is similar in part, the application to integral equations is treated differently in the two papers.

† For details, see E. Goursat, *Cours d'analyse mathématique*, Paris (1923), (third edition), Vol. 3, p. 358.

‡ We should notice in this connection, however, an important paper by G. C. Evans, *Trans. Amer. Math. Soc.*, Vol. 11 (1910), pp. 393-411.

The object of the present paper is to show how a modification of the Riemann definition of fractional differentiation and integration can be used to obtain the explicit solution of a class of integral equations of type (1). From this application some insight is gained into the connection between the singular points of differential equations and equations of type (1) where the continuity restrictions on $G(x, s)$ have been removed.

Part I is devoted to a discussion of the fractional operations which will be employed and Part II establishes the connection between these operations and integral equations.

PART I.

1. *Fractional Operations.* The history of attempts to give a logical definition of fractional differentiation and integration starts with Leibnitz and includes such names as Abel, Liouville, Riemann, Fourier, Holmgren, Hadamard, Pincherle, etc.*

J. Liouville in three long articles in the *J. éc. polyt.*, Vol. 13 (1832) gave an extensive treatment of the subject and applied his formulas to various types of mechanical problems. Liouville considered functions which can be expanded in a convergent series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}.$$

His definition of a derivative of order s (where s is any number rational, irrational or complex) by the series

$$D^s f(x) = \sum_{n=0}^{\infty} c_n a_n{}^s e^{a_n x}$$

generalizes in a natural way the ordinary derivatives of calculus, but has the obvious disadvantage that s must be restricted to values for which the series converges. We shall see later how the theory of Liouville makes connection with the theory of the present paper.

B. Riemann in a paper developed during his student days, but posthumously published, defines fractional differentiation so as to secure an analogue with Taylor's series.†

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* For a bibliography see the *Encyclopédie des Sciences Mathématiques*, T. 11, v. 5, fas. 1, "Équations et Opérations Fonctionelles," by S. Pincherle, § 8; also *Encyklopädie d. mathematischen Wissenschaften*, 11, A, 11, "Funktionaloperationen und Gleichungen" by S. Pincherle, § 7. Consult also Abel, *Werke* (1881), pp. 11-27; Art. II, 4 de l'Encyclopédie; Kelland, *Trans. Royal Soc. of Edinburgh*, Vol. 14; Heavyside, *Electromagnetic Theory*, Vol. 2, chap. 7.

† "Versuch einer allgemeinen Auffassung der Integration und Differentiation," *Gesammelte Mathematische Werke*, Leipzig (1892), pp. 353-366.

Thus, assume that a function $z(x+h)$ can be expanded in a series of the form

$$z(x+h) = \sum_{\nu=-\infty}^{\infty} k\ D_x^{\nu} z h^{\nu}$$

where the ν's are a set of numbers which differ from one another by integers and the k's are to be so chosen that, for integral values of ν the formula will become the ordinary Taylor's expansion

$$z(x+h) = \sum_{p=0}^{\infty} 1/p!\ \frac{d^p z}{dx^p}\, h^p.$$

When the further condition is added that

$$D_x^{\nu} D_x^{\mu} z(x) = D^{\nu+\mu} z(x),$$

it is found that $D_x^{\nu} z$ may be defined as follows:

1) For ν negative and z continuous between the limits x and k,

$$D_x^{\nu} z = \frac{1}{\Gamma(-\nu)} \int_c^x (x-t)^{-\nu-1} z(t)\,dt + \sum_{n=1}^{\infty} K_n \frac{x^{-\nu-n}}{\Gamma(-n-\nu+1)}$$

where K_n are finite arbitrary constants. We may designate this complementary function by ψ_{ν}.

2) For $\nu \geqq 0$,

$$D_x^{\nu} z = \frac{d^m}{dx^m} \frac{1}{\Gamma(m-\nu)} \int_c^x (x-t)^{m-\nu-1} z(t)\,dt + \frac{d^m}{dx^m}\, \psi_{\nu},$$

where m is an integer greater than ν.

The obvious difficulty with Riemann's theory, as Cayley points out,* rests with the degree of arbitrariness of the complementary function. Involving as it does an infinite number of arbitrary constants, the operational symbol is clearly not determinant and it becomes of first importance to remove this difficulty from the theory.

Hence it seems clear that any definition of fractional operations which will be of ultimate use must satsfy four conditions:

1) The operation $D_x^{\nu} z(x)$ for negative integers must reduce to ordinary integration, for positive integers to ordinary differentiation, and for $\nu = 0$ must leave the function $z(x)$ unchanged.

* Note on Riemann's paper "Versuch . . . ," *Math. Annalen*, Vol. 16.

2) We must have, when the complementary function is identically zero,

$$D_x^{\nu} D_x^{\mu} z(x) = D_x^{\nu+\mu} z(x).$$

3) The degree of arbitrariness of the complementary function must be finite and determinant.

4) The fractional operations must possess the property of linearity,

$$D_x^{\nu} [y(x) + z(x)] = D_x^{\nu} y(x) + D_x^{\nu} z(x).$$

2. *A Definition of Fractional Integration.*

Consider the iterated integral

$$I = \int_c^X dx \int_c^x (x-s)^n \phi(s)\, ds.$$

Apply to this integral the Dirichlet formula for integration over the triangle $s = c$, $s = x$, $x = X$, and we shall have

$$\begin{aligned} I &= \int_c^X ds \int_s^X (x-s)^n \phi(s)\, dx \\ &= \int_c^X \frac{(X-s)^{n+1}}{n+1} \phi(s)\, ds. \end{aligned}$$

By successive application of this formula to

$$\int_c^x \phi(s)\, ds^n = \int_c^x ds \int_c^x ds \cdots \int_c^x \phi(s)\, ds$$

it follows that

$$\int_c^x \phi(s)\, ds^n = \int_c^x \frac{(x-s)^{n-1}}{(n-1)!} \phi(s)\, ds.$$

We shall now define for all positive values of n a general symbol of integration by means of a natural generalization of this formula. Thus

$$\phi^{(-\nu)}(x) = D_x^{-\nu}\phi(x) = \int_c^x \frac{(x-s)^{\nu-1}}{\Gamma(\nu)} \phi(s)\, ds$$

where ν may run over all positive real values.

Clearly, since our extended integration is defined by means of a definite integral, it possesses the property of linearity. It is easily shown that this operation also obeys the index law.

By definition

$$D_x^{-\mu} D_x^{-\nu} \phi = \int_c^x \frac{(x-s)^{\mu-1}}{\Gamma(\mu)} \int_c^s \frac{(s-t)^{\nu-1}\phi(t)}{\Gamma(\nu)}\, dt\, ds.$$

Apply Dirichlet's formula to this integral and there results

$$D_x^{-\mu} D_x^{-\nu} \phi = \int_c^x \phi(t)\, dt \int_t^x \frac{(x-s)^{\mu-1}}{\Gamma(\mu)} \frac{(s-t)^{\nu-1}}{\Gamma(\nu)}\, ds.$$

Make the transformation

$$y = \frac{s-t}{x-t}$$

and we shall have

$$D_x^{-\mu} D_x^{-\nu} \phi = \int_c^x (x-t)^{\mu+\nu-1} \phi(t)\, dt \int_0^1 \frac{(1-y)^{\mu-1} y^{\nu-1}}{\Gamma(\mu)\, \Gamma(\nu)}\, dy$$

$$= \int_c^x \frac{(x-t)^{\mu+\nu-1}}{\Gamma(\mu+\nu)} \phi(t)\, dt.$$

But this last integral is exactly $D_x^{-(\mu+\nu)} \phi(x)$ according to our definition. We next show that

$$\lim_{\nu=0} D^{-\nu} \phi(x) = \phi(x).$$

Suppose $\phi(s)$ to be expanded into a Taylor's series with a remainder. Then

$$\lim_{\nu=0} D^{-\nu} \phi(x) = \lim_{\nu=0} \Big[\int_c^x \frac{(x-s)^{\nu-1}}{\Gamma(\nu)} \phi(x)\, ds - \int_c^x \frac{(x-s)^{\nu}}{\Gamma(\nu)} \phi'(x)\, ds$$

$$+ \cdots + (-1)^n \int_c^x \frac{(x-s)^{\nu+n-1}}{\Gamma(\nu)\, n!} \phi^{(n)}(x)\, ds$$

$$+ (-1)^{n+1} \frac{\phi^{n+1}(\xi)}{(n+1)!} \int_c^x \frac{(x-s)^{\nu+n}}{\Gamma(\nu)}\, ds.$$

Since $\lim_{\nu=0} \Gamma(\nu) = \infty$, all terms except the first vanish. Moreover

$$\lim_{\nu=0} \int_c^x \frac{(x-s)^{\nu-1}}{\Gamma(\nu)}\, ds = \lim_{\nu=0} \frac{(x-c)^{\nu}}{\nu\Gamma(\nu)} = 1,$$

from which it follows at once that

$$\lim_{\nu=0} D^{-\nu} \phi(x) = \phi(x).$$

3. *The Complementary Function and Some Formulas.*

With Riemann we may now define a generalized differentiation by means of the formula

$$\phi^{(\nu)}(x) = D_x^{\nu}(x) = \frac{d^m}{dx^m} \phi^{(\nu-m)}(x)$$

where m is an integer greater than ν. As a matter of convenience in notation we shall suppose hereafter that ν is a fraction between 0 and 1.

Then our extended definition of differentiation and integration may be written down as follows:

$$ {}_cD_x^{-(m+\nu)}\phi(x) = \int_c^x \frac{(x-s)^{m+\nu-1}}{\Gamma(m+\nu)}\,\phi(s)\,ds, $$

$$ {}_cD_x^{m+\nu}\phi(x) = \frac{d^{m+1}}{dx^{m+1}}\int_c^x \frac{(x-s)^{-\nu}}{\Gamma(1-\nu)}\,\phi(s)\,ds. $$

In general it will be necessary to indicate the lower limit of the integral in the operational symbol, because, clearly, these operations are not unique, but depend upon a single parameter. The significance of this parameter in connection with the complementary function is now to be pointed out.

We will notice first that, although the above definition, on the basis of a logical extension of formula (5) to fractional values of n, is very different from Riemann's definition by means of an infinite series, we are led virtually to the same conclusions. There is one very important difference, however, and that is in the omission of Riemann's complementary function ψ_ν with its infinite number of arbitrary constants.

The essential nature of this complementary function may be easily ascertained if we consider the difference

$$ C(x) = {}_{c'}D_x^{-\nu}\,{}_cD_x^{-\mu}\,\phi(x) - {}_cD_x^{-\nu-\mu}\,\phi(x). $$

When $c'=c$, we have already proved that $C(x)\equiv 0$, but when $c'\neq c$, this will not, in general, be true. In other words $C(x)$, the complementary function, measures the deviation of our integration symbol from the index law, this deviation depending clearly upon the choice of c'.

In order to obtain a complementary function depending upon as many arbitrary parameters as we choose, it is only necessary to proceed as follows. Let the number n be resolved into p positive fractions $n_1\cdots, n_p$ whose sum equals n. Then

$$ {}_cD_x^{-n}\,\phi = {}_{c_1}D_x^{-n_1}\,{}_cD_x^{-n+n_1}\,\phi + C_1(x) $$

$$ {}_cD_x^{-n+n_1}\,\phi = {}_{c_2}D_x^{-n_2}\,{}_cD_x^{-n+n_1+n_2}\,\phi + C_2(x) $$

$$ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot $$

$$ {}_cD_x^{-n+\sum_{i=1}^{p-1} n_i}\,\phi = {}_{c_p}D^{-n_p}\,{}_cD_x^{0}\,\phi + C_p(x). $$

The successive substitution of these symbols in the one preceding gives us finally

$$ {}_cD_x^{-n}\,\phi = {}_{c_1}D_x^{-n_1}\,{}_{c_2}D_x^{-n_2}\cdots{}_{c_p}D_x^{-n_p}\,\phi + C_1(x) + {}_{c_1}D_x^{-n_1}\,C_2(x) + $$
$$ {}_{c_1}D_x^{-n_1}\,{}_{c_2}D_x^{-n_2}\,C_3(x) + \cdots + {}_{c_1}D_x^{-n_1}\,{}_{c_2}D_x^{-n_2}\cdots{}_{c_p}D_x^{-n_p}\,C_p(x), $$

which, since each $C_i(x)$ depends upon the choice of an arbitrary c_i, is a function that depends upon p arbitrary parameters.

In all of the applications that follow we shall suppose that the same parameter is used throughout and except when there is danger of ambiguity, this limit will not be written in the symbol.

A few examples of formulas obtained by an application of these fractional operations to several of the elementary functions are appended below.

$$ {}_cD_x^{-\nu}\, k = k\,\frac{(x-c)^{\nu}}{\Gamma(1+\nu)}, \qquad {}_0D_x^{-\nu}\, x^n = \frac{\Gamma(n+1)}{\Gamma(n+\nu+1)}\, x^{n+\nu} $$

$$ {}_cD_x^{\nu}\, k = k\,\frac{(x-c)^{-\nu}}{\Gamma(1-\nu)}, \qquad {}_0D_x^{\nu}\, x^n = \frac{\Gamma(n+1)}{\Gamma(n-\nu+1)}\, x^{n-\nu} \qquad n \neq 1 $$

$$ {}_0D_x^{-\nu} \log x = \frac{1}{\Gamma(1+\nu)}\, x^{\nu}[\log x + \Gamma(\nu+1)\, T(\nu)], $$

$$ {}_0D_x^{\nu} \log x = \frac{1}{\Gamma(1-\nu)}\, x^{-\nu}[\log x + \frac{1}{1-\nu} + \Gamma(2-\nu)\, T(1-\nu)] $$

where $T(\nu) = \int_0^1 \frac{(1-t)^{\nu-1}\log t}{\Gamma(\nu)}\, dt.$

$$ {}_0D_x^{-\nu}\, e^x = e^x \int_0^x \frac{t^{\nu-1}e^{-t}}{\Gamma(\nu)}\, dt, $$

$$ {}_0D_x^{\nu}\, e^x = e^x \int_0^x \frac{t^{-\nu}e^{-t}}{\Gamma(1-\nu)}\, dt + \frac{1}{\Gamma(1-\nu)}\, x^{-\nu}\ ; $$

$$ {}_0D_x^{-\nu} \sin x = \frac{x^{\nu}}{\Gamma(2+\nu)}\ [x - \frac{x^3}{(2+\nu)(3+\nu)} + \frac{x^5}{(2+\nu)(3+\nu)(4+\nu)(5+\nu)} - \cdots] $$

$$ {}_0D_x^{\nu} \sin x = \frac{x^{-\nu}}{\Gamma(2-\nu)}\ [x - \frac{x^3}{(2-\nu)(3-\nu)} + \frac{x^5}{(2-\nu)(3-\nu)(4-\nu)(5-\nu)} - \cdots] $$

4. *Connection with the Definition of Liouville.*

Suppose, according to Liouville, that $f(x)$ can be expanded into a convergent series of the form

$$ f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}. $$

Apply the operational symbol ${}_{-\infty}D^{\nu}_{x}$ to $f(x)$ and we shall have

$$ {}_{-\infty}D_x^{\nu} f(x) = {}_{-\infty}D_x^{\nu} \sum_{n=0}^{\infty} c_n e^{a_n x} = \sum_{n=0}^{\infty} c_n \, {}_{-\infty}D_x^{\nu} e^{a_n x}. $$

This step, of course, presupposes that the operational symbol can be put under the summation sign, a thing that is far from obvious. However, since the operational symbol is essentially an integration, or an integration followed by differentiation, the usual criterion for term by term integration and differentiation will generally apply. The fact that the lower limit is negative infinity, however, requires further care, but the criterion for term by term integration between infinite limits is well known and need not be specifically recalled here.* We will suppose that the above step has been justified by some one of the tests that apply to this case.

By definition

$$ {}_{-\infty}D_x^{\nu} e^{a_n x} = \frac{d}{dx} \int_{-\infty}^{x} \frac{(x-s)^{-\nu}}{\Gamma(1-\nu)} e^{a_n s} ds. $$

If we make the transformation $s = x - t$, we shall have

$$ {}_{-\infty}D_x^{\nu} e^{a_n x} = \frac{1}{a_n{}^{1-\nu}} \frac{d}{dx} e^{a_n x} \int_0^{\infty} \frac{(at)^{-\nu} e^{-a_n t} a_n}{\Gamma(1-\nu)} dt $$

$$ = a_n{}^{\nu} e^{a_n x}. $$

Consequently Liouville's definition and the definition of this paper are identical for the particular form of the operation which results when the parameter c is taken equal to minus infinity.

PART II.

5. *Abel's Integral Equation.* With this brief exposition of a theory of fractional operations, we now turn to an application to the theory of integral equations of the Volterra type. The first suggestion of this application is to be found in Abel's works † where the following integral equations is studied:

$$ \phi(x) = \int_0^x \frac{u(s)}{(x-s)^{\nu}} ds, \qquad \nu < 1. $$

Clearly this may be written in the form

$$ \phi(x) = \Gamma(1-\nu) u^{(\nu-1)}(x). $$

Take the $1-\nu$th derivative of both sides

$$ \phi^{(1-\nu)}(x) = \Gamma(1-\nu) u(x) $$

* See Bromwich, *Infinite Series*, London (1908), p. 452.

† Abel, *loc. cit.*

from which we have at once

$$u(x)=\frac{1}{\Gamma(1-\nu)}\,\frac{1}{\Gamma(\nu)}\,\frac{d}{dx}\int_0^x\frac{\phi(s)}{(x-s)^{1-\nu}}\,ds$$

or, since $\Gamma(1-\nu)\,\Gamma(\nu)=\pi/\sin\pi\nu$,

$$u(x)=\frac{\sin\pi\nu}{\pi}\,\frac{d}{dx}\int_0^x\frac{\phi(s)}{(x-s)^{1-\nu}}\,ds.$$

6. *Some Special Cases of Volterra Equations of Second Kind.*

Before proceeding to more general considerations, it will be illuminating to study several special examples of integral equations of the Volterra type.

Example 1. Solve

$$u(x)+\phi\,(x)=\int_0^x\frac{u(s)}{(x-s)^{\frac{1}{2}}}\,ds.$$

By means of the definition and properties of fractional operations we shall have

$$\begin{aligned}
&u(x)+\phi(x)=\Gamma(\tfrac{1}{2})u^{(-\frac{1}{2})}(x),\\
&u^{(-\frac{1}{2})}(x)+\phi^{(-\frac{1}{2})}(x)=\Gamma(\tfrac{1}{2})u^{(-1)}(x),\\
&u(x)+\phi(x)=\Gamma(\tfrac{1}{2})\;[\Gamma(\tfrac{1}{2})u^{(-1)}(x)-\phi^{(-\frac{1}{2})}(x)],\\
&u'(x)-\pi u(x)=F(x),
\end{aligned}\tag{8}$$

where $F(x)=-\phi'(x)-\dfrac{d}{dx}\displaystyle\int_0^x\frac{\phi(s)}{(x-s)^{\frac{1}{2}}}\,ds.$

The solution of equation (8) is

$$u(x)=ce^{\pi x}+e^{\pi x}\int_0^x F(x)\,e^{-\pi x}dx,$$

but when this value for $u(x)$ is substituted in the integral equation it is found that, while the second term is, indeed, a solution, the first term fails to satisfy the equation. This result might have been anticipated from the discussion given in the introduction, but that this is not always the case is illustrated in the following example:

Example 2. Consider the homogeneous equation

$$u(x)=\int_{-\infty}^x\frac{u(s)}{(x-s)^{\frac{1}{2}}}\,ds.$$

Then $\qquad u(x)=\Gamma(\tfrac{1}{2})u^{(-\frac{1}{2})}(x),$

and by the same arguments used in the last example we obtain the auxiliary equation

$$u'(x) - \pi u(x) = 0.$$

The solution of this equation,

$$u(x) = ce^{\pi x},$$

when substituted in the integral equation is, indeed, found to be a solution, which would not have been anticipated from the theory given in the introduction.

Another example illustrates the connection between the order of the auxiliary equation and the degree of $x - s$.

Example 3. Solve the equation

$$u(x) + \phi(x) = \int_0^x \frac{u(s)}{(x-s)^{1/3}}\, ds.$$

Employing the properties of fractional operations, we have

$$u(x) + \phi(x) = \Gamma(\tfrac{2}{3}) u^{(-2/3)}(x),$$
$$u^{(2/3)}(x) + \phi^{(2/3)}(x) = \Gamma(\tfrac{2}{3}) u(x),$$
$$u^{(4/3)}(x) + \phi^{(4/3)}(x) = \Gamma(\tfrac{2}{3}) u^{(2/3)}(x),$$
$$u''(x) + \phi''(x) = \Gamma(\tfrac{2}{3}) u^{(4/3)}(x),$$

From these equations it follows, finally, if we set $\Gamma^{3/2}\ (2/3) = \lambda$, that the auxiliary equation will be

$$u''(x) - \lambda^2 u(x) = F(x),$$

where $F(x) = -\lambda^{4/3}\phi^{(2/3)} - \lambda^{2/3}\phi^{(4/3)} - \phi''$.

The complete solution of the differential equation will then be

$$u(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + \frac{1}{\lambda}\int_0^x \sinh \lambda\ (x-t) F(t)\, dt$$

where c_1 and c_2 are arbitrary constants.

Just as in the first example, when this function is substituted in the integral equation the last member is found to be a solution, but $u(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$ is not a solution for any values whatsoever of the arbitrary constants.

The interpretation of these results will be made in the last section of this paper.

7. *Volterra Equations with a Polynomial Kernel.*

So far we have been restricted to the consideration of a very special class of equations, but we shall now extend our results by considering the following Volterra equation

$$u(x)+\phi(x)=\int_c^x \frac{g(s)}{(x-s)^{1-a}}u(s)\,ds, \quad a>0,$$

where $g(s)$ is a polynomial of nth degree.

This equation may be written in the symbolic form

$$u(x)+\phi(x)=\Gamma(a)\,[g(x)u(x)]^{(-a)},$$

which leads us to consider the question of the fractional integration and differentiation of a product.

By definition

$$[U(x)V(x)]^{(-a)}=\int_c^x \frac{U(s)V(s)}{\Gamma(a)(x-s)^{1-a}}\,ds. \tag{9}$$

Suppose that $U(s)$ can be developed in a Taylor's series. If this expansion is substituted in (9) we shall have, providing the series to be integrated is uniformly convergent,

$$\begin{aligned}&[U(x)V(x)]^{(-a)}=\\ &U(x)\int_c^x \frac{V(s)}{\Gamma(a)(x-s)^{1-a}}\,ds-U'(x)\int_c^x \frac{V(s)}{\Gamma(a)(x-s)^{-a}}\,ds\\ &+\frac{U''(x)}{2!}\int_c^x \frac{V(s)}{\Gamma(a)(x-s)^{-(a+1)}}\,ds-\cdots=U(x)V^{(-a)}(x)\\ &-aU'(x)V^{-(a+1)}(x)+\frac{a(a+1)}{2!}U''(x)V^{-(a+2)}(x)-\cdots\end{aligned} \tag{10}$$

This formula is, of course, a generalization on the basis of our definitions, of the well known *formula of Leibnitz.**

Similarly, in order to obtain the $(m+a)$th derivative of the product $U(x)\;V(x)$, we shall have

$$[U(x)V(x)]^{(m+a)}=\frac{d^{m+1}}{dx^{m+1}}[U(x)V(x)]^{-(1-a)},$$

and if the resulting series, after it has been differentiated term by term, is uniformly convergent, this series will be the required derivative.

We can now return to the integral equation with polynomial kernel and

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* For generalizations of a similar character see Pincherle, *loc. cit.*, especially Opérations fonctionelles, etc., § 16.

apply it to the expansion (10). Since $g(s)$ is a polynomial, the series will be terminating and we shall have

$$\begin{aligned} u(x) + \phi(x) &= \Gamma(a)\,[g(x)u(x)]^{(-a)}, \\ &= \Gamma(a)\,[g(x)u^{(-a)}(x) - ag'(x)u^{-(a+1)}(x) + \cdots \\ &\quad + (-1)^n a(a+1)\cdots(a+n-1)/n!\; g^{(n)}(x)u^{-(a+n)}(x)]. \end{aligned} \tag{11}$$

By repeated differentiation and the use of methods similar to those employed in algebraic rationalization, we can convert (11) into a linear differential equation of pth order of the form

$$u^{(p)}(x) + P_1(x)u^{(p-1)}(x) + \cdots + P_n(x)u(x) = R(x) \tag{12}$$

where $P_1(x), \cdots P_n(x)$ are polynomials. In this reduction we have supposed, of course, that a is a rational fraction, because otherwise the order of equation (12) would be infinite.

An example will serve to illustrate the general method of reduction.

Example. Solve the equation

$$u(x) + \phi(x) = \int_c^x \frac{s\,u(s)}{(x-s)^{\frac{1}{2}}}\,ds.$$

Writing this equation in fractional form, we have

$$u(x) + \phi(x) = \Gamma(\tfrac{1}{2})\;[x\,u(x)]^{(-\frac{1}{2})}.$$

We may now apply the rules of fractional operation and obtain

$$\begin{aligned} u^{(\frac{1}{2})}(x) + \phi^{(\frac{1}{2})}(x) &= \Gamma(\tfrac{1}{2})\;x\,u(x) \qquad (13) \\ u'(x) + \phi'(x) &= \Gamma(\tfrac{1}{2})\;[x\,u^{(\frac{1}{2})}(x) + \tfrac{1}{2}u^{(-\frac{1}{2})}(x)] \\ u''(x) + \phi''(x) &= \Gamma(\tfrac{1}{2})\;[x\,u^{(3/2)}(x) + 3/2\,u^{(\frac{1}{2})}(x)]. \end{aligned}$$

From this equation, equation (13), and the equation obtained by differentiating (13) once, we may eliminate $u^{(1/2)}(x)$ and $u^{(3/2)}(x)$. Then we have

$$u'' - \pi x^2 u' - 5/2\,\pi x u = F(x)$$

where $F(x) = -[\sqrt{\pi}x\phi^{(3/2)} + \sqrt{\pi}\,3/2\,\phi^{(\frac{1}{2})} - \phi''].$

The solution of the integral equation may be obtained from the general integral of this equation by taking the arbitrary parameters equal to zero, as we shall prove in the next section.

8. *Singular Solutions of Volterra Equations.*

In previous sections we have shown how to associate with a Volterra equation of type (1), a linear differential equation

$$L(u) = f(x), \tag{14}$$

but it was also apparent that (1) and (14) are not necessarily equivalent since the complete integral of (14) is not, in general, a solution of (1).

The difficulty arose when we tried to solve the homogeneous equation

$$u(x) = \int_a^x \frac{g(x, s)\, u(s)}{(x-s)^\nu}\, ds \tag{15}$$

which, from the ordinary theory, will have no solution if $g(x, s)$ is a continuous function in both variables in the region $a \leqq s \leqq x \leqq b$.*

Suppose, now, that when the method of fractional operations is applied to (15) we are led to a homogeneous differential equation of nth order

$$L(u) = 0 \tag{16}$$

with coefficients analytic in a region S.

The following theorem states a necessary condition that a solution of (16) shall also be a solution of (15) when continuity restrictions have been removed from $g(x, s)$.

THEOREM. *If the x and s derivatives of $g(x, s)$ to the nth order exist and are continuous in s alone in the interval $a \leqq s \leqq x$, then a necessary condition that* (15) *shall have a continuous solution different from zero with derivatives to the nth order is that $L(u) = 0$ shall have a singular point for $x = a$.*

If we take the derivative of both sides of (15) we shall obtain the following equation:†

$$u'(x) = \frac{g(x, a)\, u(a)}{(x-a)^\nu} + \int_a^x \frac{g_x(x, s) + g_s(x, s)}{(x-s)^\nu}\, u(s)\, ds + \int_a^x \frac{g(x, s)}{(x-s)^\nu}\, u'(s)\, ds.$$

Clearly, under our hypothesis concerning $g(x, s)$, $u'(x)$ exists except for exceptional values of x. Moreover, since from (15) $u(a) = 0$, it follows that $u'(a) = 0$. By taking successive derivatives we shall have, similarly, $u''(a) = u'''(a) = \cdots = u^{(n)}(a) = 0$. But unless $L(u) = 0$ has a singu-

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* See M. Bôcher, *An Introduction to Integral Equations*, Cambridge (1909), p. 19.

† By the method of "partie finie" due simultaneously to R. D'Adhemar and J. Hadamard which extends the formula for differentiation under the sign of integration to functions defined as in (15). See Hadamard, Congrès de Mathém., 1904; D'Adhemar, *Exercices et leçons d'analyse*, Paris, (1908), pp. 150 and 180.

lar point for $x = a$, $u(x)$, from the theory of linear differential equations with analytic coefficients, must be identically zero which is contrary to our hypothesis. This contradiction proves the theorem.

The following corollary is immediately evident if we recall that all solutions of $L(u) = 0$, under the assumption that the coefficients of $L(u)$ are analytic, are single valued analytic functions except in the neighborhood of the singular points of the differential equation.

Corollary. If the x and s derivatives of $g(x, s)$ of all orders exist and *are continuous in s alone in the interval $a \leqq s \leqq x$, then a necessary condition that* (15) *shall have a continuous solution different from zero with derivatives of all orders is that $L(u) = 0$ shall have an essentially singular point for $x = a$.*

As another corollary we may state, recalling equation (12), that

When $g(x, s)$ is a polynomial in s and a is finite, equation (15) *has no solution except the trivial one.*

The second example of section 6 furnishes us with a case of a solution with an essential singularity at minus infinity. The auxiliary equation

$$u'(x) - \pi u(x) = 0$$

has $u = ce^{\pi x}$ for a solution which vanishes, together with its derivatives of all orders for $x = -\infty$.

An interesting example, showing the role played by a finite singular point, is given below.

Consider

$$u(x) = \int_0^x \frac{u(s)}{x(x-s)^{\frac{1}{2}}}\, ds. \tag{17}$$

Then, by fractional operations, we shall have

$$\begin{aligned} x\, u(x) &= \sqrt{\pi}\, u^{(-\frac{1}{2})}(x), \\ \sqrt{\pi}\, u(x) &= [x\, u(x)]^{(\frac{1}{2})}, \\ &= x\, u^{(\frac{1}{2})}(x) + \tfrac{1}{2} u^{(-\frac{1}{2})}(x), \\ &= x\, u^{(\frac{1}{2})}(x) + \frac{1}{2\sqrt{\pi}}\, x\, u(x). \end{aligned}$$

Also $$x\, u'(x) + u(x) = \sqrt{\pi}\, u^{(\frac{1}{2})}(x),$$

from which we have at last

$$x^2 u'(x) + (3/2\, x - \pi) u(x) = 0.$$

From the equation we see that $x = 0$ is a singular point and from the explicit solution,

$$u(x) = ce^{-\pi/x}\, x^{-3/2}, \tag{18}$$

it appears that the function has an essential singularity there. Moreover $u(x)$ and all of its derivatives approach the limit zero as we approach the origin along the positive real axis.

If we substitute this function in the right hand side of (17) and call the resulting integral $I(x)$ we shall have

$$I(x) = c \int_0^x \frac{s^{-3/2}\, e^{-\pi/s}}{(x-s)^{\frac{1}{2}}}\, ds.$$

Let $s = \dfrac{tx}{x+t}$, and we shall have

$$\begin{aligned} I(x) &= c\, x^{-\frac{1}{2}}\, e^{-\pi/x} \int_0^\infty e^{-\pi/t}\, t^{-3/2}\, dt \\ &= c\, x^{-\frac{1}{2}}\, e^{\pi/x}. \end{aligned}$$

Hence $I(x)$ is identical with $xu(x)$ and (18) is thus proved to be a solution of (17).

INDIANA UNIVERSITY.

Representation of Three-element Algebras.*

By B. A. Bernstein.

1. *Introduction.* I have shown in a previous paper † that the 2^4 polynomials

$$(1) \qquad (A_1 a + A_2) b + A_3 a + A_4 \pmod 2, \qquad (A_i = 0, 1),$$

where a and b range over the values 0, 1, are equivalent to the 2^4 operations $a \oplus b$ given by the tables

$$(2) \qquad \begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & e_1 & e_2 \\ 1 & e_3 & e_4 \end{array}, \qquad (e_i = 0, 1).$$

This fact enables us to obtain for a class of two elements convenient arithmetic representations not only of all class-closing operations but also, as I have shown, of all relations and of all operations that do not satisfy the condition of closure. I now wish to show that there exist for three-element algebras representations quite analogous to the arithmetic representations given for the two-element case.

2. *Representation of class-closing operations.* The arithmetic representations of binary operations that satisfy the condition of closure in a three-element algebra, are given by the following

Theorem. *The* 3^9 *polynomials*

$$(3) \qquad \begin{cases} (A_1 a^2 + A_2 a + A_3) b^2 + (A_4 a^2 + A_5 a + A_6) b \\ \qquad\qquad + A_7 a^2 + A_8 a + A_9 \pmod 3 \\ A_i = 0, 1, 2 \end{cases}$$

where a and b range over the values 0, 1, 2, *are equivalent to the* 3^9 *operations* $a \oplus b$ given by the $\oplus$-tables

$$(4) \qquad \begin{array}{c|ccc} \oplus & 0 & 1 & 2 \\ \hline 0 & e_1 & e_2 & e_3 \\ 1 & e_4 & e_5 & e_6 \\ 2 & e_7 & e_8 & e_9 \end{array} \qquad (e_i = 0, 1, 2).$$

* Read before the American Mathematical Society, December 27, 1923.

† "Complete Sets of Representations of Two-Element Algebras," *Bull. Amer. Math. Society*, Vol. XXX, p. 24.

The theorem is true because, first, every function (3) determines a $\oplus$-table (4). Secondly, as it may easily be verified, every $\oplus$-table (4) determines the A's of a function $\phi(a, b)$ of form (3) from the equations (modulo 3)

$$\begin{cases} e_1 = \phi(0,0) = A_9 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ e_9 = \phi(2,2) = A_1 + 2A_2 + A_3 + 2(A_4 + 2A_5 + A_6) + A_7 + 2A_8 + A_9. \end{cases}$$

We may call the function (3) equivalent to a given $\oplus$-table the $\oplus$-*function* of the table.

3. *Calculation of the $\oplus$-function.* If a $\oplus$-table is given, its $\oplus$-function can be obtained by the method indicated in the proof of our theorem. A more convenient method is that based on a table of functions $\phi(x)$ given by

(5) $$\phi(x) = c_1x^2 + c_2x + c_3 \pmod 3, \quad (c_i = 0, 1, 2).$$

This table follows.

No.	$\phi(0)$	$\phi(1)$	$\phi(2)$	$\phi(x) = c_1x^2 + c_2x + c_3 \pmod 3$
1	0	0	0	0
2			1	$2x^2 + x$
3			2	$x^2 + 2x$
4	0	1	0	$2x^2 + 2x$
5			1	x^2
6			2	x
7	0	2	0	$x^2 + x$
8			1	$2x$
9			2	$2x^2$
10	1	0	0	$2x^2 + 1$
11			1	$x^2 + x + 1$
12			2	$2x + 1$
13	1	1	0	$x^2 + 2x + 1$
14			1	1
15			2	$2x^2 + x + 1$
16	1	2	0	$x + 1$
17			1	$2x^2 + 2x + 1$
18			2	$x^2 + 1$
19	2	0	0	$x^2 + 2$
20			1	$x + 2$
21			2	$2x^2 + 2x + 2$
22	2	1	0	$2x + 2$

No.	$\phi(0)$	$\phi(1)$	$\phi(2)$	$\phi(x) = c_1x^2 + c_2x + c_3 \pmod 3$
23			1	$2x^2 + 2$
24			2	$x^2 + x + 2$
25	2	2	0	$2x^2 + x + 2$
26			1	$x^2 + 2x + 2$
27			2	2

Thus, let us find the $\oplus$-function $\phi(a, b)$ of the table

(i)

$\oplus$	0	1	2*
0	0	2	1
1	1	0	2
2	2	1	0.

Let

(ii) $$\phi(a, b) = \phi_1(a)b^2 + \phi_2(a)b + \phi_3(a) \quad (\text{mod } 3).$$

By our $\phi(x)$ table,

(iii) $$\phi(0, b) = 2b, \quad \phi(1, b) = 2b + 1, \quad \phi(2, b) = 2b + 2.$$

By (ii) and (iii),

(iv) $$\begin{cases} \phi_1(0) = 0, & \phi_2(0) = 2, & \phi_3(0) = 0 \\ \phi_1(1) = 0, & \phi_2(1) = 2, & \phi_3(1) = 1 \\ \phi_1(2) = 0, & \phi_2(2) = 2, & \phi_3(2) = 2. \end{cases}$$

By (iv) and the $\phi(x)$ table,

(v) $$\phi_1(a) = 0, \quad \phi_2(a) = 2, \quad \phi_3(a) = a.$$

Substituting in (ii),

(vi) $$\phi(a, b) = 2b + a \quad (\text{mod } 3).*$$

4. *Representation of dyadic relations.* Our theorem of § 2 enables us to obtain a representation of any dyadic relation aRb defined by a table of the form

(6)

R	0	1	2
0	$\pm$	$\pm$	$\pm$
1	$\pm$	$\pm$	$\pm$
2	$\pm$	$\pm$	$\pm$,

* The operation defined by (i) or (vi) is the operation of subtraction (modulo 3) among the elements 0, 1, 2.

where the sign $+$ indicates that aRb holds, the sign — that aRb does not hold. In fact, consider the $\oplus$-table obtained from a given R-table by replacing the signs $+$ by 0 and the signs — by 1. Let $\phi(a,b)$ be the $\oplus$-function of this table. Then the equation

(7) $$\phi(a,b)=0$$

will evidently be a representation of the given R-table.

We may call the equation (7) equivalent to a given R-table the *R-equation* of the table.

As an example, let us obtain the R-equation of the table

(i)

R	0	1	2*
0	—	+	+
1	—	—	+
2	—	—	—.

The table got from (i) by replacing the signs $+$ and — by 0 and 1 respectively, is

(ii)

$\oplus$	0	1	2
0	1	0	0
1	1	1	0
2	1	1	1.

The $\oplus$-function of (ii) is

(iii) $$(2a+2)b^2+(a^2+a)b+1 \pmod 3.$$

The desired R-equation of (i) is, therefore,

(iv) $$(2a+2)b^2+(a^2+a)b+1=0 \pmod 3.\text{*}$$

5. *Representation of operations not class-closing.* In a class K of elements 0, 1, 2 an operation $a\oplus b$ that does not satisfy the condition of closure, is given by a $\oplus$-table

(8)

$\oplus$	0	1	2
0	e_1	.	.
1	.	.	.
2	.	.	e_9,

where at least one of the e's is an element x not in K. The theorem of § 2 helps us obtain an arithmetic representation of such a $\oplus$-table. For suppose T

---

* Relation (i) or (iv) is the relation "less than" among the elements 0, 1, 2. Compare § 6 below.

a table of form (8). Let $\phi_1(a, b)$ be the $\oplus$-function of the table got from T by replacing every x * by some K-element, and let $\phi_2(a, b)$ be the $\oplus$-function of the table got from T by replacing every x by 0 and every non-x by a K-element not 0. Then, evidently, the desired representation of T is

$$\phi_1(a, b) + \frac{0}{\phi_2(a, b)} \quad . \tag{9}$$

Thus, an arithmetic representation of

$$\text{(i)} \qquad \begin{array}{c|ccc} & 0 & 1 & 2\ \dagger \\ \hline 0 & x & 1 & 1 \\ 1 & x & 0 & 2 \\ 2 & x & 2 & 0 \end{array}$$

is the function

$$\text{(ii)} \qquad 2ab + 1 \pmod 3 + 0/b, \ddagger$$

where

$$\text{(iii)} \qquad 2ab + 1 \pmod 3,\ b$$

are the respective $\oplus$-functions of

$$\text{(iv)} \qquad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0, \end{array} \qquad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2. \end{array}$$

6. *An application.*‡ The representations discussed above have an important bearing on the problem of proving consistency and independence of a set of postulates. Our representations not only furnish us with a host of concrete systems of the same type which may serve as the desired proof-systems, but also they often enable us to write down these system almost mechanically (since their $\oplus$-tables and R-tables can often be written down almost mechanically). I shall illustrate the last fact by discussing the

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* We may take all the x's the same.

† Operation (i) or (ii) is the operation $1 - a/b \pmod 3$. It is the operation in terms of which Wiener (*Trans. Amer. Math. Society*, Vol. 21 (1920), p. 237) defined a field. Function (ii) shows that in a three-element class this operation can be expressed directly in terms of addition and multiplication, hence ultimately in terms only of addition. Moreover since addition and multiplication can be defined in terms of Wiener's operation, all of the 3^9 operations (3) of § 2 can be expressed in terms of this operation.

‡ For other applications (confined to two-element classes) the reader is referred to my paper cited above.

complete existential theory of a set of postulates for serial order due to Huntington.*

Huntington's postulates, expressed in terms of a class K and a dyadic relation $<$, are as follows:

P_0. The class K is not empty, nor a class consisting merely of a single element.

P_1. If a and b are distinct elements of K, then either $a < b$ or $b < a$.

P_2. [If a and b are elements of K, and] † if $a < b$, then a and b are distinct.

P_3. [If a, b, c are elements of K, and] ‡ if $a < b$ and $b < c$, then $a < c$.

With regard to this postulate-set we observe at once † that there exist no systems $(K, <)$ having the *characters*

(i) $(----)$, (ii) $(---+)$, (iii) $(--+-)$,
(iv) $(--++)$, (v) $(-+--)$, (vi) $(-++-)$.

But there exist systems having the remaining ten of the 2^4 characters $(\pm\pm\pm\pm)$. The existing systems $(K, <)$ are given in the following table. In each case the smallest possible number of elements is taken for K. The existence of systems ix-xvi proves, of course, that postulates P_1, P_2, P_3 are *completely independent.*

No.	*Character*	K	$a < b$	
vii	$(-+-+)$	0	$\begin{array}{c\|c} & 0 \\ \hline 0 & + \end{array}$	$0 = 0$
viii	$(-+++)$	0	$\begin{array}{c\|c} & 0 \\ \hline 0 & - \end{array}$	$0 \neq 0$
ix	$(+---)$	0, 1, 2	$\begin{array}{c\|ccc} & 0 & 1 & 2 \\ \hline 0 & + & + & - \\ 1 & - & - & + \\ 2 & - & - & - \end{array}$	$(2a+2)b^2 + (a+1)b + a^2 = 0 \pmod 3$
x	$(+--+)$	0, 1	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & - \\ 1 & - & + \end{array}$	$a + b = 0 \pmod 2$

* E. V. Huntington, *The Continuum*, 2d ed., 1917, Harvard University Press, p. 10.

† The bracketed conditions are mine. They are obviously implied by Professor Huntington.

‡ Since the denial of P_0 implies P_1 and P_2.

No.	*Character*	K	$a < b$	
xi	$(+-+-)$	$0, 1, 2$	$\begin{array}{c\|ccc} & 0 & 1 & 2 \\ \hline 0 & - & + & - \\ 1 & + & - & - \\ 2 & - & - & - \end{array}$	$(a^2+2a+1)b^2+(2a^2+1)b$ $+a^2+a+1=0 \pmod 3$
xii	$(+-++)$	$0, 1$	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & - \\ 1 & - & - \end{array}$	$1=0$
xiii	$(++--)$	$0, 1$	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & + & - \end{array}$	$ab=0 \pmod 2$
xiv	$(++-+)$	$0, 1$	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & + & + \\ 1 & - & - \end{array}$	$a=0$
xv	$(+++-)$	$0, 1$	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & + & - \end{array}$	$a+b+1=0 \pmod 2$
xvi	$(++++)$	$0, 1$	$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & - & + \\ 1 & - & - \end{array}$	$ab+b+1=0 \pmod 2$

7. *Note on the representation of $n(>3)$-element algebras.* We naturally ask: Is there a function of degree $n-1$, analogous to (3) of § 2, for the class-closing operations in an algebra having $n>3$ elements? The question must be answered in the negative. There is no such function for $n=4$. But there is one for $n=5$. The complete theory of modular representation for the general case I hope to discuss in a future paper.

UNIVERSITY OF CALIFORNIA,
January, 1924.

The Riemann Adjoints of Completely Integrable Systems of Partial Differential Equations*

By Cyril A. Nelson.

1. *Introduction.*

From a geometrical point of view, the most important property of the Lagrange adjoint of a linear homogeneous, ordinary differential equation is that its solutions correspond, by the principle of duality, to the solutions of the equation itself. When one passes to completely integrable systems of linear homogeneous partial differential equations of the second order in two independent variables, this property has been the determining factor in generalizing the adjoint concept. There is, however, another point of view which one might adopt, namely, one might generalize the mode of derivation of the Lagrange adjoint rather than its geometrical property. This was the method adopted by Riemann for a single partial differential equation and has proved valuable in the integration theory of the equation.

This paper is devoted to the investigation of questions which arise when one considers the system of equations formed by the Riemann adjoints of a completely integrable system.

2. *Formulation of the Problem.*

Wilczynski has shown that every completely integrable system of linear homogeneous partial differential equations of the second order in two independent variables, whose integral surfaces are not developables, can be reduced to the canonical form

$$\begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0, \end{aligned} \tag{1}$$

where $y_{uu} = \partial^2 y/\partial u^2$, etc. If the coefficients of this equation are analytic functions of u and v and the integrability conditions

$$\begin{aligned} &g_u + a'_{uu} + 2ba'_v + 4a'b_v = 0, \\ &f_v + b_{vv} + 2a'b_u + 4ba'_u = 0, \\ &g_{uu} + 4gb_v + 2bg_v = f_{vv} + 4fa'_u + 2a'f_u, \end{aligned} \tag{2}$$

are identically satisfied the system (1) defines four linearly independent solutions $y^{(k)}$, $(k = 1, 2, 3, 4)$. Moreover the most general solution of (1) is

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* Presented to the American Mathematical Society, February 24, 1923.

expressible linearly with constant coefficients in terms of $y^{(k)}$. The four linearly independent solutions may be called the homogeneous coördinates of a point in ordinary space. As u and v vary over their range, the point describes a surface, the integral surface S_y of (1). The parametric curves on this surface are the asymptotic curves.

The homogeneous coördinates $Y^{(k)}$ of the tangent planes of S_y are linearly independent solutions of a completely integrable system similar to (1), namely,

$$(3) \qquad \begin{aligned} Y_{uu} - 2bY_v + (f + 2b_v)Y = 0, \\ Y_{vv} - 2a'Y_u + (g + 2a'_u)Y = 0. \end{aligned}$$

The system (3) is called adjoint to (1) and may be regarded as the generalization of the Lagrange adjoint, in the sense that the fundamental geometric property of the latter is preserved. The integrability conditions of (3) are, of course, equivalent to (2).*

Consider any equation of the form

$$Ay_{uu} + By_{uv} + Cy_{vv} + Dy_u + Ey_v + Fy = 0.$$

The adjoint of this equation, in the sense of Riemann,† is

$$(Az)_{uu} + (Bz)_{uv} + (Cz)_{vv} - (Dz)_u - (Ez)_v + Fz = 0.$$

Accordingly, the Riemann adjoints of the equations of system (3) are

$$(4) \qquad \begin{aligned} Z_{uu} + 2bZ_v + (f + 4b_v)Z = 0, \\ Z_{vv} + 2a'Z_u + (f + 4a'_u)Z = 0. \end{aligned}$$

We shall see later that the system formed by the Riemann adjoints of the equations of a completely integrable system are not, in general, completely integrable. There is, however, a relation between the solutions of (1), (3), and (4) which we shall point out. Consider any solution $\bar{y}$ of the first equation of system (1), so that

$$\bar{y}_{uu} + 2b\bar{y}_v + f\bar{y} = 0.$$

If $\bar{y}$ is placed in the second equation of (1) we have

$$\psi = \bar{y}_{vv} + 2a'\bar{y}_u + g\bar{y}$$

and ψ will not be identically zero unless $\bar{y}$ happens to be a solution of the

* E. J. Wilczynski, "Projective Differential Geometry of Curved Surfaces," *Transactions of the American Mathematical Society*, Vol. 8 (1907), pp. 233-260. Cited as *First Memoir*.

† Cf. Darboux, *Théorie des Surfaces*, t. 2, chap. 4.

system (1). However ψ is always a solution of the first equation of (4). In fact, direct computation shows

$$\begin{aligned}
\psi_v &= \bar{y}_{vvv} + 2a'\bar{y}_{uv} + 2a'_v\bar{y}_u + g\bar{y}_v + g_v\bar{y}, \\
\psi_{uu} &= -2b\bar{y}_{vvv} - (f + 4b_v)\,\bar{y}_{vv} - 4a'b\bar{y}_{uv} + 2(g_u + a'_{uu} - a'f)\bar{y}_u \\
&\quad - [2b_{vv} + 2f_v + 4a'b_u + 2b(g + 4a'_u)]\bar{y}_v \\
&\qquad - [f_{vv} + 2a'f_u - g_{uu} + f(g + 4a'_u)]\bar{y},
\end{aligned}$$

whence, on account of (2),

$$\psi_{uu} + 2b\psi_v + (f + 4b_v)\psi = 0.$$

In the same way, we find that every solution $\bar{y}$ of the second of system (1) gives rise to the function

$$\phi = \bar{y}_{uu} + 2b\bar{y}_v + f\bar{y}$$

which is a solution of the second equation of (4).

We have said that the two equations (4) do not form, in general, a completely integrable system. In the following sections we determine all systems (1) whose Riemann adjoints possess this property.

3. *Analytic Solution of the Problem when $a'b \neq 0$.*

If the coefficient b in equations (1) is identically zero, the first member shows that, for $v =$ const., y is a solution of an ordinary differential equation of the second order. The general solution is, then, a linear combination with constant coefficients, of two independent ones. Hence the parametric curves $v =$ const. are straight lines and the integral surface is ruled. Similarly, if $a' = 0$, the $u =$ const. curves are rectilinear. Throughout this section we assume that the integrating surface of (1) is not ruled so that $a'b \neq 0$.

The Riemann adjoints of the two equations of (1) are

$$\begin{aligned}
z_{uu} - 2bz_v + (f - 2b_v)z &= 0, \\
z_{vv} - 2a'z_u + (g - 2a'_u)z &= 0.
\end{aligned} \tag{5}$$

In order that the equations (5) form a completely integrable system it is necessary and sufficient that the coefficients satisfy conditions similar to (2). These may be written down immediately by replacing in (2), b, f, a', g by $-b, f - 2b_v, -a', g - 2a'_u$, respectively. The result is

$$\begin{aligned}
-a'_{uu} + g_u - 2a'_{uu} + 2ba'_v + 4a'b_v &= 0, \\
-b_{vv} + f_v - 2b_{vv} + 2a'b_u + 4ba'_u &= 0. \\
g_{uu} - 2a'_{uuu} - 4b_v(g - 2a'_u) - 2b(g_v - 2a'_{uv}) & \\
= f_{vv} - 2b_{vvv} - 4a'_u(f - 2b_v) - 2a'(f_u - 2b_{uv}). &
\end{aligned} \tag{6}$$

When we bear in mind that the equations (2) are identically satisfied, we are able to simplify the above equations considerably. In fact, the following are equivalent to (2) and (6)

$$
(7)\qquad \begin{aligned}
&g_u + 2ba'_v + 4a'b_v = 0, \quad f_v + 2a'b_u + 4ba'_u = 0,\\
&bg_v + 2b_v g = a'f_u + 2a'_u f, \quad a'_{uu} = 0, \quad b_{vv} = 0,\\
&ba'_{uv} - a'b_{uv} = 0.
\end{aligned}
$$

Consequently, *the equations* (1) *and their Riemann adjoints form completely integrable systems if, and only if, the coefficients of* (1) *satisfy* (7).

We proceed to the integration of (7). The first, second, and last of equations (7) imply $g_{uu} = f_{vv}$. Hence there exists a function ω of u and v such that $f = \omega_{uu}$ and $g = \omega_{vv}$. Moreover $ba'_{uv} = a'b_{uv}$ teaches us that functions $\lambda(u, v)$ and $\mu(u, v)$ exist which satisfy the equalities.

$$
ba'_u = \lambda_u, \quad a'b_v = \lambda_v, \quad ba'_v = \mu_v, \quad a'b_u = \mu_u.
$$

It follows that the first two equations of (7) may be replaced by

$$
\omega_{uv} = k - 4\lambda - 2\mu,
$$

k being an arbitrary constant. We may now write the conditions upon $a', b, f,$ and g in the form

$$
(7')\qquad \omega_{uuu} = \alpha\omega_{vvv} + \beta\omega_{uu} + \gamma\omega_{vv}, \quad \omega_{uv} = \delta,
$$

$$
(7'')\qquad a'_{uu} = 0, \quad b_{vv} = 0, \quad ba'_{uv} - a'b_{uv} = 0,
$$

where we have placed

$$
(8)\qquad \alpha = b/a', \quad \beta = -2\, a'_u/a', \quad \gamma = 2\, b_v/a', \quad \delta = k - 4\lambda - 2\mu.
$$

For the moment we suppose that a' and b are known functions of u and v. Then the two conditions on ω will be consistent if, and only if, the integrability conditions of (7′) are satisfied. We obtain these by computing the values of $\omega_{uuuu}, \omega_{uuuv}, \omega_{uuvv}, \omega_{uvvv}, \omega_{vvvv}$ from (7′). They are the unique expressions

$$
(9)\qquad \begin{aligned}
&\omega_{uuuu} = \alpha^{(1)}\omega_{vvv} + \beta^{(1)}\omega_{uu} + \gamma^{(1)}\omega_{vv} + \delta^{(1)},\\
&\omega_{uuuv} = \delta_{uu}, \quad \omega_{uuvv} = \delta_{uv}, \quad \omega_{uvvv} = \delta_{vv},\\
&\omega_{vvvv} = \alpha^{(2)}\omega_{vvv} + \beta^{(2)}\omega_{uu} + \gamma^{(2)}\omega_{vv} + \delta^{(2)},
\end{aligned}
$$

where

$$
(10)\qquad \begin{aligned}
&\alpha^{(1)} = \alpha_u + \alpha\beta, \quad \beta^{(1)} = \beta_u + \beta^2, \quad \gamma^{(1)} = \gamma_u + \beta\gamma,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \delta^{(1)} = \alpha\delta_{vv} + \gamma\delta_v,\\
&\alpha^{(2)} = -1/\alpha\ (\gamma + \alpha_v), \quad \beta^{(2)} = -\beta_v/\alpha, \quad \gamma^{(2)} = -\gamma_v/\alpha,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \delta^{(2)} = 1/\alpha\ (\delta_{uu} - \beta\delta_u).
\end{aligned}
$$

The derivatives of ω of the fifth order are easily calculated. However, there is a lack of uniqueness since ω_{uuuuv}, for example, may be obtained in two ways, namely, from $\partial/\partial v\ \omega_{uuuu}$ and $\partial/\partial u\ \omega_{uuuv}$. The same remark applies to ω_{uuuvv}, ω_{uuvvv}, and ω_{uvvvv}. It is apparent that of the conditions for uniqueness

$$\partial/\partial v\ \omega_{uuuu} = \partial/\partial u\ \omega_{uuuv}, \quad \partial/\partial v\ \omega_{uuuv} = \partial/\partial u\ \omega_{uuvv},$$
$$\partial/\partial v\ \omega_{uuvv} = \partial/\partial u\ \omega_{uvvv}, \quad \partial/\partial v\ \omega_{uvvv} = \partial/\partial u\ \omega_{vvv},$$

the second and third are identically satisfied since all the derivatives involved are formed from $\omega_{uv} = \delta$. It can be shown—by direct computation if we wish—that the first and last conditions are equivalent. We use the first one and find, by equating the coefficients of ω_{vvv}, ω_{uu}, ω_{vv}, and 1 in the two members of $\partial/\partial v\ \omega_{uuuu} = \partial/\partial u\ \omega_{uuuv}$, that

$$(11)\quad \begin{array}{l} \alpha^{(1)}\alpha^{(2)} + \alpha_v{}^{(1)} + \gamma^{(1)} = 0,\ \alpha^{(1)}\beta^{(2)} + \beta_v{}^{(1)} = 0, \alpha^{(1)}\gamma^{(2)} + \gamma_v{}^{(1)} = 0, \\ \alpha^{(1)}\delta^{(2)} + \beta^{(1)}\delta_u + \delta^{(1)}{}_v - \delta_{uuu} = 0. \end{array}$$

These are the integrability conditions which the coefficients must satisfy if (7′) is to be consistent in ω. If the first two be expressed in terms of a' and b through (10) and (8) we obtain the result

$$(12)\qquad \frac{b}{a'}\frac{\partial^2}{\partial u\partial v}\log\frac{b}{a'} = 0, \quad \frac{\partial \log a'b}{\partial u}\frac{\partial^2 \log a'}{\partial u\partial v} - \frac{\partial^3 \log a'}{\partial u^2\partial v} = 0.$$

These equations, together with $a'_{uu} = 0$ from (7″), prove that

$$\frac{\partial^2 \log a'}{\partial u\partial v} = \frac{\partial^2 \log b}{\partial u\partial v} = 0.$$

Let the variables y, u, and v be replaced by the new variables

$$(\mathrm{T})\qquad \bar{y} = \frac{1}{\lambda(u,v)}\, y, \quad \bar{u} = \phi(u), \quad \bar{v} = \psi(v)\ ,$$

where λ, ϕ, ψ, satisfy the condition $\lambda = \dfrac{\text{const.}}{\sqrt{\phi_u\psi_v}}$. The canonical system (1) is unaltered in form and the coefficients $\bar{a}'$ and $\bar{b}$ of the system which replaces (1) are given by

$$(13)\qquad \bar{a}' = \frac{\phi_u}{\psi_v{}^2}\, a', \quad \bar{b} = \frac{\psi_v}{\phi_u{}^2}\, b.^*$$

Hence $\overline{\left(\dfrac{b}{a'}\right)} = \left(\dfrac{\psi_v}{\phi_u}\right)^3 \dfrac{b}{a'}$ and since $\dfrac{\partial^2 \log a'}{\partial u\partial v} = \dfrac{\partial^2 \log b}{\partial u\partial v} = 0$, ϕ and ψ may be chosen so as to make $\bar{b} = \bar{a}'$. We suppose that this has been done and place $a' = b = \kappa$. This simplification replaces the two equations in (12) by

* Wilczynski, *First Memoir*, pp. 256, 247, 249.

the single condition $\dfrac{\partial^2 \log \kappa}{\partial u \partial v} = 0$. A simple computation now shows that the last two conditions of (11), when expressed in terms of $a' = b = \kappa$, are verified. We summarize our results as follows: *If the Riemann adjoints of the two members of any completely integrable system of homogeneous, linear, partial differential equations in one dependent and two independent variables, whose integral surfaces are not ruled, form a completely integrable system, the original equations may be made to assume the form*

$$y_{uu} + 2\kappa y_v + fy = 0, \quad y_{vv} + 2\kappa y_u + gy = 0, \tag{14}$$

with

$$g_u + 6\kappa\kappa_v = 0, \quad f_v + 6\kappa\kappa_u = 0, \quad \kappa f_u + 2\kappa_u f = \kappa g_v + 2\kappa_v g, \tag{15}$$

$$\kappa_{uu} = \kappa_{vv} = \frac{\partial^2 \log \kappa}{\partial u \partial v} = 0.$$

Any transformation (T) that does not violate the conditions

$$a' = b, \qquad \lambda = \frac{\text{const.}}{\sqrt{\phi_u \psi_v}}$$

may be used to simplify (14) and (15). The equation (13) shows that any transformation for which $\phi_u{}^3 = \psi_v{}^3 = \text{const.}$ will preserve the form (14). The last three equations of (15) are satisfied in the most general manner by

$$\kappa = (a_1 u + a_2)(a_3 v + a_4)$$

in which a_1, a_2, a_3, a_4 are arbitrary constants. It is convenient to distinguish four cases.

a) $a_1 a_3 \neq 0$. The transformation $\bar{u} = (a_1{}^{-2} a_3)^{1/3} (a_1 u + a_2)$, $\bar{v} = (a_1 a_3{}^{-2})^{1/3} (a_3 v + a_4)$ makes $\bar{\kappa} = \bar{u}\bar{v}$. We suppose that this change of variables has been effected. The first three equations of (15) may then be easily integrated with the result

$$y_{uu} + 2uvy_v + fy = 0, \quad y_{vv} + 2uvy_u + gy = 0, \tag{14a}$$

with

$$f = ku - u^4 + k_1 u^{-2} - 2uv^3, \quad g = kv - v^4 + k_2 v^{-2} - 2u^3 v, \tag{15a}$$

in which k, k_1, k_2 are arbitrary constants.

b) $a_1 \neq 0$, $a_3 = 0$. In this case $\bar{\kappa}$ can be reduced to $\bar{u}$ by the transformation $\bar{u} = (a_1{}^{-1} a^4)^{\frac{1}{3}} (a_1 u + a_2)$, $\bar{v} = (a_1 a_4)^{\frac{1}{3}} v$. After the strokes have been dropped and f, g found by integrating (15) we have

$$y_{uu} + 2uy_v + fy = 0, \quad y_{vv} + 2uy_u + gy = 0, \tag{14b}$$

with

$$f = ku + k_1u^{-2} - 6uv, \quad g = 3kv - 9v^2 + k_2. \tag{15b}$$

c) $a_1 = 0$, $a_3 \neq 0$. We interchange u and v, f and g in b) and find

$$y_{uu} + 2vy_v + fy = 0, \quad y_{vv} + 2vy_u + gy = 0, \tag{14c}$$

with

$$f = 3ku - 9u^2 + k_1, \quad g = kv + k_2v^{-2} - 6uv. \tag{15c}$$

d) $a_1 = a_3 = 0$. Any transformation for which $\phi_u = \psi_v = a_2a_4$ makes $\bar{\kappa} = 1$. We drop the strokes and write

$$y_{uu} + 2y_v + fy = 0, \quad y_{vv} + 2y_u + gy = 0, \tag{14d}$$

with

$$f = c_0u + c_1, \quad g = c_0v + c_2. \tag{15d}$$

The last theorem may now be restated with equations (14), (15) replaced by (14a), (15a); (14b), (15b); (14c), (15c); (14d), (15d). The corresponding integral surfaces shall be called solutions of type I, II, III, IV, respectively.

4. *The Non-Ruled Surfaces of the Problem.*

In order to investigate the geometrical properties of the surfaces defined in the previous section, it is desirable to recall some results due to Wilczynski.* Upon an integral surface S of (1) consider the two-parameter family of *hypergeodesics* defined by

$$v'' = A(v')^3 + 3B(v')^2 + 3Cv' + D, \quad v' = \frac{dv}{du}, \quad v'' = \frac{d^2v}{du^2}, \tag{16}$$

where A, B, C, D are analytic functions of u and v. A point P of S and a direction at P determine a unique curve. The osculating planes of the hypergeodesics through P envelop the *axis-plane cone* of class three and order four. This cone has the tangent plane π at P as a double plane and possesses three cuspidal planes which have the *cusp-axis* in common. As P assumes all positions on S, these cusp-axes form a two-parameter family of lines protruding

* E. J. Wilczynski, "Some Generalizations of Geodesics," *Transactions of the American Mathematical Society*, Vol. 23 (1922), pp. 223-239. Cited as *Geodesic Paper*.

from the surface, the *cusp-axis congruence.* This congruence cuts S in two one-parameter families of *cusp-axis* curves whose differential equations are

$$(17)\qquad \begin{aligned}&(f + 2b_v - 3bB + \tfrac{9}{4}C^2 - \tfrac{3}{2}C_u)\,du^2 - \tfrac{3}{2}(B_u + C_v)\,du\,dv\\ &\qquad -(g + 2a'_u + 3a'C + \tfrac{9}{4}B^2 + \tfrac{3}{2}B_v)\,dv^2 = 0.\end{aligned}$$

The dual of the osculating plane of a given hypergeodesic at P is the Laplace transform of P in the direction conjugate to the direction of the given hypergeodesic. The totality of these points, for all directions at P, lie on a plane cubic curve (flex-point cubic) of class four with a double point at P. The three flex-points of this cubic lie on the *flex-ray,* a line in the tangent plane π but not passing through P. If the flex-rays be constructed for all points of S, the *flex-ray congruence* is formed. The differential equations of the *flex-ray curves* are

$$(18)\qquad \begin{aligned}&(f + 3bB + \tfrac{9}{4}C^2 - \tfrac{3}{2}C_u)\,du^2 - \tfrac{3}{2}(B_u + C_v)\,du\,dv\\ &\qquad -(g - 3a'C + \tfrac{9}{4}B^2 + \tfrac{3}{2}B_v)\,dv^2 = 0.\end{aligned}$$

The cusp-axis and flex-ray for P are Green reciprocals of each other in the sense that these lines are polar reciprocals with respect to the quadric osculating S at P.

If $A = D = 0$ in (16), the cuspidal planes of the axis-plane cone cut the tangent plane π in the three Segre tangents of P. Also, in this case, the three flex-points of the flex-point cubic lie on the Darboux tangents of P. The differential equations of the Segre and Darboux *curves* are

$$(19)\qquad a'dv^3 - b\,du^3 = 0,$$

$$(20)\qquad a'dv^3 + b\,du^3 = 0,$$

respectively.

Let us also recall the definition of the *directrix curves* of S. The two linear complexes that osculate S along the two asymptotics at P have a linear congruence in common. One of these directrices, d', of this congruence passes through P and protrudes from S while the other, d, lies in the tangent plane π and does not pass through P. The lines d and d' are the directrices of the first and second kind, respectively.* The totality of lines d' form a congruence which meets S in the directrix curves. The congruence of lines d deter-

* E. J. Wilczynski, "Projective Differential Geometry of Curved Surfaces," *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 114.

mines the same curves. Indeed, Green * has shown that this property is characteristic of directrix curves provided the corresponding lines of the congruences are Green reciprocals. Using this theorem, we may find the differential equations of the directrix curves by placing the coefficients of (17) and (18) proportional to each other, obtaining a result which is equivalent to

$$bLdu^2 + 2Mdudv - a'Ndv^2 = 0, \tag{21}$$

since $a'b \neq 0$. We have used the abbreviations

$$\begin{aligned} L &= -2a'(2a'bf + 2a'bb_v + ba'_{uu}) + ba'_u{}^2, \\ M &= a'^2b^2\ \partial^2/\partial u\partial v \log b/a', \\ N &= -2b(2a'bg + 2a'ba'_u + a'b_{vv}) + a'b_v{}^2. \end{aligned} \tag{22}$$

We are now in a position to deduce some geometrical properties of the non-ruled surfaces defined by (14). For these surfaces $a = b = \bar{\kappa}$. Whence $M = 0$ by (22). But if $M = 0$ the harmonic invariant of (21) and $dudv = 0$ vanishes. Since $dudv = 0$ defines the asymptotic curves on S, it follows that *the directrix curves, on all the surfaces defined by* (14) *form conjugate systems.*

The Segre and Darboux curves, (19) and (20), may be grouped into pairs to form three conjugate systems whose differential equations are of the form

$$dv^2 - \lambda^2 du^2 = 0. \tag{23}$$

where λ^2 has the values $-(b/a')^{2/3}$, $-\omega(b/a')^{2/3}$, or $-\omega^2(b/a')^{2/3}$, ω being in imaginary cube root of unity. When $a' = b = \kappa$ these *Segre-Darboux curves* are isothermally conjugate. To prove this in the most direct fashion it is necessary to refer the surface S to the Segre-Darboux curves by means of the change of parameters $\bar{u} = \phi(u, v)$, $\bar{v} = \psi(u, v)$ where ϕ and ψ are solutions of

$$\theta_u{}^2 + \lambda^2\theta_v{}^2 = 0.$$

Instead of making the necessary computations we use the general formulas computed by Lane.† Since the Segre-Darboux curves are conjugate the system replacing (1) is of the form

$$y_{\bar{u}\bar{u}} = \bar{a}y_{\bar{v}\bar{v}} + \bar{b}\, y_{\bar{u}} + \bar{c}y_{\bar{v}} + \bar{d}y, \quad y_{\bar{u}\bar{v}} = \bar{b}'y_{\bar{u}} + \bar{c}'y_{\bar{v}} + \bar{d}'y,$$

* G. M. Green, "Memoir on the General Theory of Surfaces and Rectilinear Congruences," *Transactions of the American Mathematical Society*, Vol. 20 (1919), p. 92.

† E. P. Lane, "A General Theory of Conjugate Nets," *Transactions of the American Mathematical Society*, Vol. 23 (1922). The formulas used are found on pp. 285, 286, and 288.

in which $\bar{a} = -(\psi_v/\phi_v)^2$. The conjugate system is isothermally conjugate when $\dfrac{\partial^2 \log \bar{a}}{\partial \bar{u} \partial \bar{v}} = 0$. In terms of the asymptotic parameters this becomes

$$(24) \qquad \frac{\partial^2 \log \bar{a}}{\partial \bar{u} \partial \bar{v}} = -\frac{1}{\lambda \phi_v \psi_v} \frac{\partial^2 \log \lambda}{\partial u \partial v}.$$

If we place, in (24), the value of λ belonging to the Segre-Darboux curves we find

$$(24') \qquad \frac{\partial^2 \log \bar{a}}{\partial \bar{u} \partial \bar{v}} = -\frac{1}{3} \frac{1}{\lambda \phi_v \psi_v} \frac{\partial^2}{\partial u \partial v} \log \frac{b}{a'}.$$

Since $a' = b = \kappa$, $\dfrac{\partial^2 \log \bar{a}}{\partial \bar{u} \partial \bar{v}} = 0$ and the Segre-Darboux curves are isothermally conjugate. Consequently, *all the non-ruled surfaces which give solutions to the problem are cut in conjugate systems by their directrix congruences.* These surfaces are further restricted by the conditions.

$$\kappa_{uu} = \kappa_{vv} = \frac{\partial^2 \log \kappa}{\partial u \partial v} = 0.$$

Comparing (24′) with the expression for M in (22) we may state the more general theorem: *A necessary and sufficient condition that the Segre-Darboux curves be isothermally conjugate is that the directrix curves form a conjugate system.*

5. *The Special Case* $\kappa = 1$.

When $\kappa = 1$, so that (14) takes the simple form (14d), the surfaces defined may be completely characterized by geometric properties. In the first place, these are the surfaces whose invariants of lowest order vanish. Their properties have been discussed in detail elsewhere.* We propose to give a new characterization of these surfaces. For this purpose we introduce an invariant integral which Fubini † suggested as a projective substitute for the integral giving the length of arc. This integral is

$$I = \int \sqrt{a'bv'}\, du$$

and its extremals are

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* E. J. Wilczynski, "On a Certain Class of Self-Projective Surfaces," *Transactions of the American Mathematical Society*, Vol. 14 (1913), pp. 421-443. Also, "On a Certain Completely Integrable System of Linear Partial Differential Equations," *American Journal of Mathematics*, Vol. 36 (1914), pp. 231-260.

† G. Fubini, "Fondamenti della geometria proiettivo-differenziale di una superficie," *Atti della Reale Accademia delle Scienze di Torino*, Vol. 53 (1918), p. 1033.

$$(25) \qquad v'' = -\frac{(a'b)_v}{a'b}(v')^2 + \frac{(a'b)_u}{a'b} v'.*$$

This equation is of the form (16). The cusp-axis and flex-ray curves defined for (25) by (17) and (18) coincide if, and only if,

$$(26) \qquad -\frac{(a'b)_v}{a'b} = \frac{b_v}{b}, \quad \frac{(a'b)_u}{a'b} = -\frac{a'_u}{a'}$$

But these conditions are equivalent to $a'b^2 = U(u)$, $a'^2b = V(v)$, where U and V are arbitrary functions of their arguments. From (13) we infer that a transformation (T) can be found which reduces both $a'b^2$ and a'^2b to unity. We may therefore assume $a'^3 = b^3 = 1$ and, without loss of generality, $a = b = 1$. If these values of a' and b are placed in (2) we find the system (1) must be of the form (14d). Conversely, if $a' = b = 1$ (26) is satisfied. Hence, *a necessary and sufficient condition that a surface S be a solution of type IV is that the cusp-axis and flex-ray curves, defined with respect to the extremals of Fubini's integral invariant, coincide.*

6. *The Ruled Surfaces of the Problem.*

In the previous sections we assumed that $a'b$ was different from zero. If $a' = 0$ and $b \neq 0$, the integral surfaces of (1) are ruled. The same is true when $a' \neq 0$ and $b = 0$. If both a' and b are zero, the surfaces are ruled in two ways and hence are quadrics. In this case the integrability conditions (2) reduce to

$$(25) \qquad f_v = 0, \quad g_u = 0,$$

and the conditions that the Riemann adjoints form a completely integrable system, namely (6), become identical with (25). Hence, *the Riemann adjoints of the equations of the completely integrable systems which characterize quadrics also form completely integrable systems.*

It remains to discuss the case $a' = 0$, $b \neq 0$ or $a' \neq 0$, $b = 0$. We take the former. For this, system (1) reduces to

$$(26) \qquad \begin{aligned} &y_{uu} + 2by_v + fy = 0, \\ &y_{vv} + gy = 0, \end{aligned}$$

and the equations (2) and (6) are equivalent to

$$(27) \qquad g_u = 0, \quad f_v = 0, \quad bg_v + 2b_vg = 0, \quad b_{vv} = 0.$$

In order to investigate the properties of the ruled integrating surfaces of (26) it is convenient to set up the system of ordinary differential equations which

* Cf. Wilczynski, *Geodesic Paper*, p. 234.

characterize these surfaces. For this purpose we place $y_v = z$ and find that y, z satisfy the following

$$(28) \qquad \begin{aligned} &y_{uu} + fy + 2bz = 0, \\ &z_{uu} - 2bgy + (f + 2b_v)z = 0. \end{aligned}$$

As v remains fixed and u varies the points P_y and P_z trace out curves upon the ruled surface S_y and the line joining the points P_y, P_z generates the integrating ruled surface.* For (28) we have

$$(29) \qquad \begin{aligned} &u_{11} = -4f, \quad u_{12} = -8b, \quad u_{21} = 8bg, \quad u_{22} = -4(f + 2b_v), \\ &v_{11} = -8f_u \quad v_{12} = -16b_u, \quad v_{21} = 16b_u g, \quad v_{22} = -8(f_u + 2b_{uv}), \\ &w_{11} = -16f_{uu}, \quad w_{12} = -32b_{uu}, \quad w_{21} = 32b_{uu}g, \quad w_{22} = -16(f_{uu} + 2b_{uuv}), \end{aligned}$$

and consequently

$$(30) \qquad \begin{aligned} &\theta_4 = 2^6(b_v^2 - 4b^2g), \quad \theta_{10} = 2^{14}g(b_u b_v - bb_{uv})^2, \\ &\theta_9 = -2^{12} \begin{vmatrix} b_v & b & bg \\ b_{uv} & b_u & b_u g \\ b_{uuv} & b_{uu} & b_{uu}g \end{vmatrix} \dagger \end{aligned}$$

If g is not zero, θ_9 vanishes. More particularly, $g_u = 0$ and $bg_v + 2b_v g = 0$ imply that all minors of the second order of θ_9 are zero. Also $\theta_{10} = 0$ and $\theta_4 \neq 0$ unless $b_v^2 - 4b^2g = 0$. In this case b_v/b could be eliminated from the preceding equation and the third of (27) giving, after a simple integration,

$$g = \frac{1}{(k \pm 2v)^2}, \quad k \text{ an arbitrary constant.}$$

Hence, *the generators of the integral ruled surfaces belong to a linear congruence whose directrices are distinct or coincident according as g is different from or equal to* $\dfrac{1}{(k \pm 2v)^2}$.‡

When $g = 0$, a glance at (27) shows that b is of the form

$$b = U_1 v + U_2,$$

where U_1 and U_2 are arbitrary functions of u alone. Although $\theta_9 = 0$, all the minors of the second order do not vanish unless the Wronskian of U_1 and U_2 is zero. Moreover $\theta_{10} = 0$ if, and only if, this Wronskian vanishes. Fin-

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* E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, 1906, p. 126 et seq.

† Ibid. The formulas here used are given, respectively, on pp. 96, 99, 101, 104, 112, 113.

‡ Ibid. p. 170. All of the statements immediately following are substantiated by this same reference.

ally, $\theta_4 \neq 0$ unless U_1 is zero. Hence, if the Wronskian of U_1 and U_2 does not vanish, the ruled surface belongs to a special linear complex and hence has only one straight line directrix. If the Wronskian vanishes the ruled surface possesses two coincident straight line directrices.

If $U_1 = 0$ all the minors of the second order of θ_9 are zero, and, accordingly, the surface belongs to a linear congruence with coincident directrices.

We may recapitulate the results as follows: *If the Riemann adjoints of the completely integrable system* (26) *form a completely integrable system the integral surfaces belong to one of the following classes:*

a) *ruled surfaces possessing two straight line directrices, distinct or coincident;*

b) *ruled surfaces having only one straight line directrix;*

c) *quadric surfaces.*

The surfaces of the first class belong to a linear congruence with distinct or coincident generators, while those of the second type belong to a special linear complex.

Conversely, every ruled surface possessing at least one straight line directrix is the integral surface of a system of partial differential equations whose Riemann adjoints form a completely integrable system. In fact, let the straight line directrix be one of the curves of reference for the ruled surface S. Let the second curve of reference be an asymptotic curve C, which is not a flecnode curve. Then

$$p_{21} = u_{21} = q_{21} = p_{12} = 0, \qquad u_{12} \neq 0, \quad q_{12} \neq 0.\text{*}$$

We may call the independent variable u. If we multiply η and ζ by properly chosen functions of u, p_{11} and p_{22} may be made zero. With these simplifications the system of equations for η and ζ become

$$\begin{aligned} \eta_{uu} + q_{11}\eta + q_{12}\zeta &= 0, \\ \zeta_{uu} + q_{22}\zeta &= 0. \end{aligned} \tag{31}$$

A transformation of the form

$$\eta = \alpha y + \beta z, \quad \zeta = \delta z, \tag{32}$$

where α, β, δ are functions of v alone, does not disturb the form of (31) for we find it is replaced by

$$\begin{aligned} y_{uu} + q_{11}y + 1/\alpha\,[\beta(q_{11} - q_{22}) + \delta q_{12}]z &= 0, \\ z_{uu} + q_{22}z &= 0. \end{aligned} \tag{33}$$

The point z, for a properly chosen function $\kappa(v)$, may be represented by

* Ibid. pp. 96, 142, 150.

(34) $$z = y_v + \kappa y.$$

We seek to determine $\alpha, \beta, \delta, \kappa$ as functions of v such that (32) and (34) are consistent and also that the equations obtained by eliminating z from (33) and (34) shall form a system of the required type, (26), whose coefficients verify the conditions (27). For brevity, call the coefficient of z in the first of (33) $\bar{q}_{12}$. Eliminating z from (33) and (34) we obtain

(35) $$y_{uu} + \bar{q}_{12} y_v + (q_{11} + \kappa \bar{q}_{12}) y = 0,$$
$$\bar{q}_{12} y_{vv} + (q_{11} - q_{22} + 2\kappa \bar{q}_{12} + \frac{\partial \bar{q}_{12}}{\partial v}) y_v$$
$$+ [\kappa (q_{11} + q_{22}) + (\kappa^2 + \kappa_v) \bar{q}_{12} + \kappa \frac{\partial \bar{q}_{12}}{\partial v}] y = 0.$$

Since (32) and (34) must be consistent

(36) $$\kappa = \frac{(\alpha\delta)_v}{\alpha\delta} - \frac{\beta_v}{\beta} - \frac{\alpha}{\beta} = \frac{\alpha_v}{\alpha}.$$

Moreover, in order to identify (35) with (26) we must place the coefficient of y_v in (35) equal to zero. Thus, beside (36), κ, α, β must satisfy

(37) $$\beta_v - \beta \frac{\alpha_v}{\alpha} + 2\kappa\beta + \alpha = 0, \quad \delta_v - \delta \frac{\alpha_v}{\alpha} + 2\kappa\delta = 0.$$

These two sets of equations are equivalent to

(38) $$\kappa = \frac{\alpha_v}{\alpha}, \quad \frac{\delta_v}{\delta} = - \frac{\alpha_v}{\alpha}, \quad \beta_v + \beta \frac{\alpha_v}{\alpha} + \alpha = 0.$$

A glance at the second of (35) shows that the evanescence of the coefficients of y_v reduces the coefficient of y to $(\kappa_v - \kappa^2)\bar{q}_{12}$. We may assume $\bar{q}_{12} \neq 0$ for otherwise the asymptotic curve of reference, C_y, would be rectilinear contrary to hypothesis and hence write (35) in the simple form

(35′) $$\begin{aligned} &y_{uu} + \bar{q}_{12} y_v + (\bar{q}_{11} + \kappa \bar{q}_{12}) y = 0, \\ &y_{vv} + (\kappa_v - \kappa^2) y = 0, \end{aligned}$$

where $\bar{q}_{12}$ is the coefficient of z in the first of (33) and $\alpha, \beta, \delta, \kappa$ satisfy (38).

The system (35′) will be completely integrable and the equations of the Riemann adjoints will possess the same property if, and only if, the conditions (27), written for (35′), are verified, in addition to (38). This can be accomplished in the simplest manner by taking

$$\alpha = p, \quad \beta = q - pv, \quad \delta = r, \quad \kappa = 0,$$

where p, q, r are arbitrary constants. The converse proof is now complete.

THE JOHNS HOPKINS UNIVERSITY,
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Further Types of Involutorial Transformations which Leave Each Cubic Surface of a Web Invariant.

By Virgil Snyder.

The purpose of this paper is to discuss some further types of involutorial birational transformations in three way space which leave each surface of a web of cubics invariant. A number of types have already been considered * but in all of them the genus of the curves invariant in the involution is 1, in accordance with the Riemann-Roch theorem for surfaces.† However, if the basis points are not chosen independently, other types appear, which have not been considered in a systematic classification. In a paper having for its purpose a different problem, Pieri ‡ introduces a (1, 2) correspondence and incidentally mentions some of the properties of the associated involution. The properties of this involution are of sufficient importance to warrant a brief summary of the way in which it is defined.

Consider the web of cubic surfaces having for basis elements a space quartic curve g_4 of genus 1, and four points. Three surfaces of the web intersect in three variable points, but if the basis points $A_1, \cdots, A_4$ are chosen in the same plane π, these and the four points P_i in which π meets g_4 make eight points through which pass all the cubic curves, section of the surfaces of the web on π, hence another basis point A_5, exists, and the surfaces intersect in two variable points. If the surfaces of the web are now associated projectively with the planes of a second space (x'), we have a

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* The non-singular webs of genus 1 have been determined by Sharpe and Snyder, "Certain Types of Involutorial Space Transformations" (second paper), *Transactions American Math. Society*, Vol. 21 (1920), pp. 52-78. See pp. 55-58. The monoidali ones by D. Montesano, "Su le trasformazioni involutorie monoidali," *Istituto Lombardo Rendiconti*, series 2, Vol. 21 (1888), pp. 579-594. One case also by Francesco Romano, "Sopra una trasformazione doppia del terzo ordine nei punti dello spazio," Catania, 1906. There are six singular webs, two of which give rise to monoidal involutions, and four-nonmonoidal. Both of the former and two of the latter have been derived by Miss Clara Moffa, "Su alcune corrispondenze birazionali involutorie dello spazio dotate di un sistema lineare di dimensione tre die superficie del terzo ordine unite." Naples, 1923 (Tesi di laurea, 1921).

† See Sharpe and Snyder, l. c., p. 54.

‡ M. Pieri, "Sulle tangenti triple di alcune superficie del sest' ordine," *Atti di Torino*, Vol. 24 (1888-9), pp. 514-526.

(1, 2) correspondence between it and (x). To the lines c' of (x') correspond quintic curves of genus 2, passing simply through each A_i and meeting g_4 in 8 points.

In the general case the involution is of order 11 and the quartic is a five-fold basis curve, thus belonging as a particular type to an important category determined by Montesano.*

The various cases in which g_4 is composite are treated, and it is mentioned that three other cases exist, not included in the preceding category, which together probably exhaust the webs of cubics of genus 2.

These three cases will be discussed in the present paper, and their position in the categories of known types will be specified. They are interesting because they furnish simple illustrations of fundamental elements having exceptional character, viz.: fundamental points having irrational curves for images, and a family of parasitic curves having a straight line in (x') for image. All three types can be reduced to the monoidal form.

1. *Basis curves a conic and a line.* Let a web of cubic surfaces F be determined by a conic g_2, a straight line m not meeting g_2, and by five points A_i.

Any three cubics of the system intersect in three variable points. But if A_5 is chosen on the quadric H determind by g_2, and the other basis points, then the system has another basis point A_6 also on H. These points and the ponts P_1, P_2 in which m meets H_2 are eight associated points, basis of a net of quadrics. By putting $F_i = x'_i$ we have a (1, 2) correspondence between the points of (x') and of (x).

2. To the lines c'_1 of (x') correspond the system of c_6, genus 2, each having 6 points on g_2, and 4 points on m, and passing through all A_i; but but among these are ∞^3 which consist of a quartic k_4 on H_2 and of a conic in a plane through m. Since every quartic meets every cubic of the web in the six points A_i, in P_1, P_2 and in four points on g_2, it is parasitic, and any surface of the web which passes through one point of the quartic must contain the whole curve. Hence the image of this quartic is a point of (x'), and the locus of the image point, as the k_4 describes H_2 is a straight line h'.

The image of a line of (x) is a cubic curve in (x') having h' for bisecant, and of a plane of (x) is a sextic surface s'_6 having h' for four-fold line, since the plane in (x) meets k_4 in four points. A plane through h' has

* D. Montesano, "Su una classe di trasformazioni razionali ed involutorie dello spazio di genere arbitrario n e di grado $2n+1$," *Giornale di Matematiche,* Vol. 31 (1892), 15 pages.

for image in (x) the quadric H_2 and a plane of the pencil m. Both images of a point of (x') are in the same plane through m. A straight line meeting h' has for image the quartic k_4 and a conic in a plane through m. The conic meets g_2 in two points.

3. Let m meet the plane f of g_2 in D. The pencil D, f is parasitic, as a cubic of the web which contains a point on f not D nor on g_2 must contain the whole line joining it to D. The image of each line is a point in (x'), and the locus of the point is a line d'.

4. A pencil of cubics of the web is determined by the plane f and the pencil of quadrics through m and the points A_i. These quadrics have as basis quartic the line m and a space cubic curve r_3 passing through the points A_i. A point of d' has two images, one a line of D, f and the other a point R of r_3.

5. The jacobian of the web of cubics consists of the quadric H_2, the plane f and the surface K_5 of coincidences, which contains g_2 simply and m triply. Through each of the points A_i pass two lines a_i, b_i which meet m and g_2. These twelve lines are also parasitic, and lie on K_5. The image of K_5 in (x') is the surface of branch points L'_6. It contains h' four-fold. The images of the lines a_i, b_i are points A'_i, B'_i in (x'), double on L'_6. The line joining A'_i, B'_i meets h' and lies on L'_6, and the plane $h'\,A'_i\,B'_i$ is tangent to L'_6 along the whole line.

6. The s'_6 images of the planes of (x) contain h' four-fold, d' simply, pass simply through each of the points A'_i, B'_i and touch L'_6 at every non-basal common point.

The c'_3 images of the lines of (x) have h' for bisecant, meet d' once and touch L'_6 in five variable points. A c'_3 and an s'_6 meet in 9 points not on h' nor d', hence the associated involution of conjugate points in (x) is of order 8.

7. The surfaces L'_6, K_5 are in $(1, 1)$ point correspondence. The plane sections of the latter have for images on the former the curve of contact of L'_6 with the image s'_6 of the given plane; they are of order 10 and genus 3, pass through all the 12 points A'_i, B'_i and meet the line h' in 5 points.

The image of a point of m is a straight line in (x'). When the point describes m the image line describes a ruled surface M'_4 of order 4, since a c_6, imags of c'_1 in (x') meets m in 4 points. Each line is a tritangent of L'_6. To a line of (x) meeting m corresponds a conic and the tritangent.

8. A point of g_2 has a line for image in (x'). As c_6 meets g_2 in 6 points the image line describes a ruled surface G' of order 6 passing through h' four times, d' twice, through the twelve points A'_i, B'_i, and containing three double generators, those through the points of contact of d' and L'_6. These points have for images in (x) the three lines l of D, f which meet the points of r_3 in f.

The complete image in (x) of G'_6 is the conic g_2 and a surface G of order 8 containing m to multiplicity 6, g_2 simply, r doubly, the 12 lines a_i, b_i simply, and the lines l doubly. It is a rational ruled surface. Similarly, the complete image of M'_4 in (x) is the line m and a surface M of order 7 containing m five-fold, r_3 simply, g_2 simply, the lines l, a, b all simply.

The complete image of h' consists of H, of G_8 and of the six planes (m, A_i). The complete image of d' is f and r_3.

9. In the involution I of conjugate points in (x), the image of an arbitrary plane is $G_8 : m^6 g_2 \cdot 12 a_i \cdot 3l \cdot r_3$. It meets K_5 in the plane s_1, K_5; the residual intersection with its conjugate s_1 consists of three lines through the point in which s_1 meets m. These lines have for images in (x') double lines of the image s'_6, all of which meet h'.

A line has for conjugate a c_8 having 8 points on g_2 and seven points on m and passing through each point A_i. Every plane through m remains invariant.* The lines of (x) which meet m and g_2 are transformed into straight lines of (x') which meet h' and are tangent to L'_6. The conjugate in (x) is another line meeting g_2 and m.

Now consider a line of (x') which meets h' and d'. Its image in (x) is a quartic k_4 of H_2, a line of the pencil D, f, and a proper line meeting m and r. Since the image in (x') of this latter line is of order two, it must be the given line counted twice. Hence we have the following theorem:

THEOREM: *The two images in (x) of an arbitrary point in (x') are on a transversal of m and r_3.*

The congruence of secants of m and r_3 is left invariant in I; every line of the congruence contains an infinite number of pairs of conjugate points.

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* Since the images of planes are surfaces of order 8 with a fixed six-fold line, this involution is included among those treated by Montesano, "Su le trasformazioni involutorie dello spazio nelle quali ai piani corrispondono superficie di ordine n con una retta $(n-2)$ pla," *Rendiconti dei Lincei*, Vol. 52 (1889), pp. 123-130. Since every line joining a pair of conjugate points meets m, it belongs in another category also treated by Montesano, "Su le trasformazioni involutorie dello spazio che deterninano un complesso lineare di rette," *Rend. Lincei*, Vol. 4 (1888), pp. 207-215 and 277-285.

10. We now have the following simple construction for the conjugate of any point P in I. The plane Pm meets K_5 in a conic k_2, which passes through two points G_1, G_2 of g_2. The plane meets r_3 in R, not on k_2.

The point in which PR meets the polar of P as to k_2 is the required conjugate.

The image of m in this plane is a conic circumscribing the triangle RG_1G_2; this conic lies on M_7.

The image of G_1 is the line G_1R. This line lies on G_8, on which r_3 is therefore a double curve.

11. The involution may be expressed in the following scheme.

$$\begin{aligned} s_1 &\sim s_8\ (m^6\ g\ \ 3l\ \ 12a_i\ r_3) \\ g &\sim G_8\ (m^6\ g\ \ 3l_2\ 12a\ r_3^2) \\ m &\sim M_7(m^5\ g\ \ 3l\ \ 12a\ \ r_3). \\ K_5 &\sim K_5\ (m^3\ g\ \ 3l\ \ 12a) \\ A_i &\sim (m, A_i) \end{aligned}$$

The residual intersection (s_8, G_8) consists of two lines, images of the points in which the conjugate s_1 meets g_2. G_8 and K_5 touch along g_2; M_7 and K_5 have common tangent planes along the three sheets of K_5 in points of m. An s_8 and M_7 meet in a vairable curve of order 6, image of a generator of M'_4.

$$J_{28}(m^{23}g_2^{\,3}\,3l^3\,12a^3\,r^3) = G_8\cdot M_7\cdot 6(m, A_i)^2\cdot \mathrm{f}.$$

Numerous special cases exist when the points A_i are restricted in position. In particular, if all five are chosen in one plane, then A_6 is in the same plane, and the quadric H consists of this plane and the plane of the conic g_2. The curve now coincides with m, and the congruence of lines joining conjugate points in I is a special linear congruence. The line d' in (x') coincides with h', so that the system s'_6 all have contact along one of the sheets through h'. In each plan through m, the involution I is the perspective Jonquieres of order two; the vertex is on the line m, and the conic of invariant points is the section of the plane with K_5.

12. If we now transform this involution through the quadratic inversion having A_i for vertex and the conic g_2 for fundamental conic, the line m is transformed into a second conic meeting the first in two points and passing through A_i, the quadric is transformed into a plane meeting each conic in two points and containing the other points A_k, and the web of cubic surfaces is transformed into a web of cubic surfaces with the transformed basis elements and a variable curve of order 5 and genus 2. Hence our transforma-

tion is equivalent to a particular case of that defined by a web of cubic surfaces having for basis elements an elliptic quartic and five coplanar basis points.

13. *Basis elements two skew lines, double point on one.* Pieri * mentions that another similar case is furnished by the cubic surfaces having two arbitrary skew basis lines, a double point on one of them, and six arbitrary basis points.

Let the line u and the six basis points A_i determine a quadric H, and let the other basis line m meet H in P_1 and P_2. One cubic of the web is composed of H and the plane π through m, meeting u in the common double point P. A pencil of cubics in the web is determined by any two points. If these are chosen on the generator g_1 of H through P_1 which meets u, one cubic of the pencil is πH. Let F_1 be another. The intersection F_1, H consists of u, g_1, and an elliptic quartic curve C_{P_1} through P, P_2 and all A_i. Similarly, let C_{P_2} be the quartic on the cubic F_2 associated with g_2 through P_2. These two quartics intersect in an eighth point Q.

Since the line m is a bisecant of a quartic C on H through P and all A_i, all the cubics of the net determined by πH, F_1, and F_2 pass through C. Let F be a cubic of the web, not in this net. It intersects H in a quintic having one further intersection Q with C. Hence Q is now a basis point of the web, and any three surfaces belonging to it intersect in two variable points.

If we now put

$$x'_1 = \pi H, \quad x'_2 = F_1, \quad x'_3 = F_2, \quad x'_4 = F,$$

we have a $(1, 2)$ point correspondence between the spaces (x'), (x). A quadric $\overline{H}$ can be passed through m and k_4, meeting u in P and in one other point T, hence a pencil of cubics in the web is composed of $\overline{H}$ and a plane of the pencil through u. A cubic of the web intersects $\overline{H}$ in a quintic curve having a double point at P, passing through all the basis points A_i, through T and meeting m in three points. These quintics are all parasitic, and the image of each is a point H'. As the quintic describes $\overline{H}$, the image point describes a straight line h'.

The lines of the pencil π, P are parasitic. The image of each is a point in (x'). As all the lines lie on π, a component of a cubic of the web, the locus of the image points is a plane curve. Since π meets any cubic of the web in two lines of the pencil, any plane of (x') meets the image locus in two points; that is, the image of the pencil π, P is a conic γ'_2.

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* l. c. Same foot-note.

The image of the quartic k_4 is a point P' on h' and on γ'_2. The planes through u cut from H the generators which meet u. Their images are the lines of the plane $x' = 0$ through the point P'. The residual point on γ'_2 is the image of the line of π, P which meets the corresponding generator of H. The image in (x') of a plane of (x) is a surface of order 7, containing h' as a five-fold line, and passing simply through γ'_2. A line of (x) goes into a cubic curve meeting h' twice and γ'_2 once.

Each point A_i goes into a plane α'_i through h'; the two parasitic lines through A_i are of different natures; the one, a_i passes through P, and the other meets u and m. These lines all go into points common to all the s'_7 and double on the surface of branch-points L'.

The point P goes into a cubic surface having h' for double line, and passing through γ'_2. A point of u goes into a straight line which describes a ruled surface of order 5, passing through γ'_2 and having h' for a triple line. Similarly, the image of m is a ruled surface M' of order 4, through γ'_2 simply and h' triply.

The surface of branch-points L' is of order 6, and contains h' as a four-fold line. A c'_3 meets it in five points of contact not on h', hence the surface K of coincidences in (x) is of order 5. It is a component of the jacobian of the web of cubics, the others being the plane π and the quadric $\overline{H}$. Since each cubic of the web has the symbol $s_3(P^2um7A_i)$, it follows that K_5 has the form $K_5(P^4u^3m7A_i7a_i7b_i)$.

To get the complete image of γ'_2 in (x), pass a quadric through γ'_2 and h'. Its image in (x) consists of $\overline{H}$ and a cubic surface having u for double line. The residual intersection with H is a rational quartic r_4 meeting u in 3 points, not passing through P, and meeting π in 4 points. The lines of π, P which connect these points with P all lie on K_5.

14. *The involution I.* We can now write the following table which completely defines the involution.

$$
\begin{array}{rl}
s_1 \sim s_9 & (P^8u^7\, m\; 7A^2{}_i\; r_4\; 4l_i\; 7a_i\; 7b_i) \\
P \sim \Pi_4 & (P^4u^3 \quad 7A_i\; r_4\; 4l_i\; 7a_i \qquad) \\
m \sim M_5 & (P^4u^4\, m\; 7A_i\; r_4\; 4l_i \qquad 7b_i) \\
u \sim U_8 & (P^7u^6\, m\; 7A^2{}_i\; r_4\; 4l_i\; 7a_i\; 7b_i) \\
r_4 \sim \pi & (P \qquad m \qquad\qquad 4l_i \qquad\qquad) \\
A_i \sim \alpha_1 & (P\, u \qquad A_i \qquad\qquad\quad a_i\;\; b_i) \\
K_5 \equiv K_5 & (P\; u\;\; m\; 7A_i \qquad 4l_i\; 7a_i\; 7b_i)
\end{array}
$$

$$J_{32}(P^{30}u^{27}m^3 7A^6{}_i\; r^3{}_4\; 4l^4{}_i\; 7a^4{}_i\; 7b^4{}_i) = \Pi_4 M_5 U_8 \pi\; 7\alpha^2{}_i$$

U_8, K_5 touch along u, three sheets.

M_5 and K_5 touch along m.

15. The congruence of lines in (x') meeting γ'_2 and h' goes into the congruence of lines meeting u and r_4. Each line of the congruence contains an infinite number of pairs of conjugate points.

16. This web can be reduced to a web of cubics through a skew quadrilateral and five coplanar basis points by means of a general (2, 3) birational transformation, hence it is equivalent to a form included among those considered by Pieri.

If we map the homaloidal web of quadrics $s_2(m\,P\,A_6, A_7)$ on the planes of an auxiliary space (x'), such that the planes of (x) go into cubics $s'_3(l'^2\,m'\,m'_6\,m'_7)$, in which the line l' meets each of the mutually skew lines m', m'_6, m'_7 then we have

$$s_1(PA_6A_7) \sim l',\quad s_1(mP) \sim m',\quad s_1(mA_6) \sim m'_6,\quad s_1(mA_7) \sim m'_7$$
$$m \sim F'_2(l'\,m'\,m'_6\,m'_7),\quad P \sim \pi'(l'm'),\quad A_6 \sim \pi'_6(l'm'_6),\quad A_7 \sim \pi'_7(l'm'_7)$$
$$\overline{H}_2(mP\,A_6\,A_7\,5A_i) \sim p'_i(5A'_i)$$

A line u through P goes into a line u'. Since each plane mA_6, mA_7 meets u in a point, u' meets m'_6 and m'_7.

Hence we can now write

$$s_3(m\,u\,P^2\,A_6\,A_7\,5A_i) \sim s'_3(l'\,u'\,m'_6\,m'_7\,5A'_i).$$

The pencil of elliptic quintics on $\overline{H}_2$ go into plane cubics through the five basis points A'_1, through T' on u', and through the points on l', m'_6, m'_7 which are images of the generators of $\overline{H}_2$ through P, A, A which meet m.*

17. *Basis curves a plane cubic and a line meeting it.* Consider a web of cubic surfaces having for basis elements a plane cubic curve g_3 of genus 1 in the plane $x_4 = 0$, a straight line m or $x_1 = 0$, $x_2 = 0$ meeting g_3, and four points A_i in the plane $x_3 = 0$. This plane meets m in a point $D = (0, 0, 0, 1)$, and meets g_3 in three collinear points P_i on the line $x_3 = 0$, $x_4 = 0$. The plane $x_3 = 0$ meets the surfaces of the web in cubic curves through the eight points D, P_i, A_i, hence they all pass through a ninth fixed point A_5. The cubic curves of this pencil are parasitic, since they have nine fixed points in common with every cubic surface of the web. The residual

* I wish to express my obligation to Dr. Jesse O. Osborn of the University of Iowa, for confirming all the above results, partly by different methods. The substance of § 16 is entirely due to him.

curve of intersection of any two surfaces of the web is of order 5 and of genus 2; it meets g_3 in 5 points, and meets m in 3 points.

The points A_i and D lie on a conic r_2; through r_2 and m can be passed a net of quadric surfaces, which with the plane $x_4=0$ provide three independent cubics of the web.

The image of the plane $x_4=0$ is a point P' in (x'). Contained within the net of quadrics through m and r_2 is a pencil consisting of the plane $x_3=0$ of the conic r_2, and of a variable plane through m. We may therefore write

$$x'_1=x_1x_3x_4,\quad x'_2=x_2x_3x_4,\quad x'_3=H_2x_4,\quad x'_4=F_3.$$
$$s'_1\sim s_3(m\ g_3\ 5A_i).$$

The Jacobian of the web is of the form

$$J_8=x_3x^2_4\,K_5,$$

in which $K_5\ :\ m^3g_3A_ia_ib_i$ is the proper surfaces of coincidences. For an arbitrary line c'_1 of (x') we have

$$c'_1\sim c_5,\quad p=2;\quad [c_5,g_3]=5;\quad [c_5,m]=3.$$

The images of the plane cubics in $x_3=0$ are points D' of a line d'. The plane $x_4=0$ has for image the point $P'\equiv(0,0,0,1)'$ on the line d' or $x'_1=0$, $x'_2=0$. The points of m have straight lines for images, which generate a ruled surface $M'_3\ :\ d'\,P'^2$. The image of any point of g_3 is a line of the bundle P', and the lines generate a quintic cone $G'_5\ :\ P'^5\,d'^3$ with two double generators, images of the lines l in $x_4=0$ which join m to the points on r_2.

$$s_1\sim s'_5(d'^3\,D'^4)\,;\ c_1\sim c'_3,\,p=0.\ [c'_3,d']=2,\ \ [c'_3,P']=1.$$

There are two lines a_i, b_i through each point A_i; the images of these lines are points in (x'), double on L'_6, and the points go into planes through d', touching L'_6.

$$K_5\sim L'_6\ (d'^4\,P'^4).$$

18. *The involution I.* The image of $s'_5\ :\ d'^3\,P'^4$ in (x) is of order $15-3-4-1=7$, hence $s_1\sim s_7\ (m^5g\ 10a_i\ r_2\ 2l)$

$$\begin{aligned}&g\sim G_7\ (m^5g_3\ 10a_i,\,r^2_2,\,2l^2),\\&m\sim M_6\ (m^4g_3\ 10a_i,\,r_2,\,2l\ ),\\&r_2\sim x_4\ \ (g_3)\\&J_{24}\equiv G_7\,M_6\,5a^2_i\,x_3. \qquad a_i\equiv s\ :\ mA_i.\end{aligned}$$

The bundle of lines (x') through P' go into the congruence of lines which meet m and r_2; every one of these lines contains an infinite number of pairs of conjugate points. The cone G'_5 has for complete image in (x) the cubic g_3 and the ruled surface defined by the transversals of g_3, r_2 and m. The two generators in the plane of g_3 meet g_3 twice, hence are double.

19. We now have a simple method of constructing the conjugate of any point P in I. Pass a plane through P and m. It will meet r_2 in a point R, and cut from K_5 a conic in which the polar of R is the line of the plane in $x_4 = 0$, which meets g_3 in two points G_1, G_2, lying on K_5. The point on PR in which it meets the polar of P as to the conic on K_5 is the required conjugate. In particular, every point of G_1G_2 has R for image, hence the conjugate of the lines of the pencil $m, x_4 = 0$ are the points of r_2. It should be noticed that the multiplicity of P' and L'_6 is no higher than any other point of d', and that L'_6 is similar to the L'_6 in the general case discussed by Pieri.

20. These $(1, 2)$ correspondences present three types of parasitic curves, that is, curves which are not basis curves of the invariant surfaces of the web, but have all their intersections with these surfaces at fixed basis points.

(a). The lines a, b are isolated rational parasitic curves. Their images in the involution are the same lines, in the sense that if a curve c meets one of them, say a, in a point P, the conjugate of c is composite, one component being a, and the other is a variable curve meeting a in P and a second point which varies with the tangent to c at P.

(b). All the lines of a pencil are parasitic. The images of these lines are points, which describe a rational curve. Any variable curve must meet some line of the pencil; its conjugate passes through the image points in a definite direction, depending on the point of intersection of the given curve and the line of the pencil.

(c). The quartic curves k_4 on H, or their transforms into plane cubic curves of a pencil. These curves are of genus 1. Let P be any point in the plane of the pencil. Through P passes one cubic c_3 of the pencil, and through c_3 pass three independent cubic surfaces of the web. In (x') the image planes determine a point P' which is the same for every point P on c_3. Any line in (x') which pass through P' has for image in (x) a composite image consisting of c_3 and of a variable conic meeting c_3 in two points forming a rational $g^1{}_2$ on c_3. The pencil of conics through P thus meet c_3 in the conjugate of P in this series. The jacobian group of the series consists of four

points on K. As c_3 describes the pencil this jacobian group describes the section of K by the plane of the pencil. There is no image curve like that of case (b). Any curve passing through P, no matter how, will have as conjugate a curve passing through the conjugate of P in the linear series on c_3.* Moreover, this case presents a simply infinite system of parasitic curves, the images of which in (x') are points of a line, and not a single point, as previously supposed.† The types suggested by Pieri are reduced to types belonging to a rational congruence ‡ and are consequently reducible to the monoidal form.

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* For the proper interpretation of this case I wish to acknowledge the assistance of Professor F. R. Sharpe.

† R. De Paolis, "Sulle trasformazioni doppie dello spazio," *Memorie della R. Acc. dei Lincei*, Vol. 1 (1885), pp. 568-609.

‡ R. De Paolis, "Alcune trasformazioni involutorie dello spazio," Nota I, *Rendiconti della R. Acc. dei Lincei*, series 4, Vol. 1 (1885), pp. 735-742. See p. 739.